Sample Test 3
Solutions

1 $P_n(x) = \sum_{i=0}^{n} L_{n,i}(x) f_i$

where (1) $L_{n,i}(x)$ is defined on p.344

(2) $f_i = f(x_i)$

$L_{3,0} = \frac{(x-2)(x-4)(x-6)}{(1-2)(1-4)(1-6)} = \frac{-1}{15}\left(x^3-12x^2+44x-48\right)$

$L_{3,1} = \frac{(x-1)(x-4)(x-6)}{(2-1)(2-4)(2-6)} = \frac{1}{8}\left(x^3-11x^2+34x-24\right)$

$L_{3,2} = \frac{(x-1)(x-2)(x-6)}{(4-1)(4-2)(4-6)} = -\frac{1}{12}\left(x^3-9x^2+20x-12\right)$

$L_{3,3} = \frac{(x-1)(x-2)(x-4)}{(6-1)(6-2)(6-4)} = \frac{1}{40}\left(x^3-7x^2+14x-8\right)$

$\Rightarrow \sum_{i=1}^{3} L_{n,i}(x) f_i =$

$\overset{-1/5}{\cancel{\frac{-3}{15}}}\left(x^2-12x^2+44x-48\right) + \overset{1/2}{\cancel{\frac{4}{8}}}\left(x^3-11x^2+34x-24\right)$

$\overset{-1/3}{\cancel{-\frac{4}{12}}}\left(x^3-9x^2+20x-12\right) + \overset{1/20}{\cancel{\frac{2}{40}}}\left(x^3-7x^2+14x-8\right)$

$= \boxed{\frac{1}{60}x^3 - \frac{9}{20}x^2 + \frac{67}{30}x + \frac{6}{5}}$

2) Recall the theorem on page 148

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\dots(x-x_n)$$

error term

interpolating polynomial

$\boxed{\text{in problem 1, } n=3}$

$$E = \frac{f^4(\xi)}{4!}(x-1)(x-2)(x-4)(x-6)$$

$\boxed{\text{also not } f'(4)(x) = f^{(4)}(x) \Rightarrow \text{typo}}$

this is bounded by .012 (given)

$$|E(5)| = \left|\frac{.012}{4!}(5-1)(5-2)(5-4)(5-6)\right|$$

take absolute value

$= .006$

ERROR BOUND

3) Divided Difference Table

x_i	f_i	1ST	2nd	3RD
0	6			
		-5		
2	-4		4	
		11		1
4	18		10	
		51		
6	120			

(x^2-6x+8)

$$P_n = 6 - 5(x-0) + 4(x-0)(x-2) + 1(x-0)(x-2)(x-4)$$

$$= 6 - 5x + 4x^2 - 8x + x^3 - 6x^2 + 8x$$

$\boxed{P_n(x) = x^3 - 2x^2 - 5x + 6} \Rightarrow \boxed{P_n(2.5) = -3.375}$

4) Theorem p. 348

$$\Rightarrow |E| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\ \cdots\ (x-x_n) \right|$$

doesn't exactly state this, but this is the general idea

Theorem p. 370

$$f[x_0, x_1 \ldots x_n] = \frac{f^{(n)}(\xi)}{n!}$$

last divided difference

Two good theorems to know about interpolating polynomial errors.

+

5) we write 4 equations as

$$a_0 + a_1 x_0 + a_2 x_0^2 + a_3 x_0^3 = f(x_0)$$
$$a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3 = f(x_1)$$
$$a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3 = f(x_2)$$
$$a_0 + a_1 x_3 + a_2 x_3^2 + a_3 x_3^3 = f(x_3)$$

Write in matrix form

note: It's assumed here that the following data is given $\Rightarrow$

x_0	$f(x_0)$
x_1	$f(x_1)$
x_2	$f(x_2)$
x_3	$f(x_3)$

$$V \Rightarrow \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix} \Rightarrow Va = b$$

$$\Rightarrow \boxed{a = V^{-1} b}$$

note: In Problem 5, V is ~~in~~ a matrix Form known as the Vandermonde matrix

6) I don't expect you to know this unless you studied it in linear algebra...

$$\det(V) = \prod_{0 \le i < j \le n} (x_j - x_i)$$

Vandermonde matrix

$\Rightarrow$ as long as $\{x_0, x_1, \dots x_n\}$ are distinct, ~~then~~ $\det(V) \neq 0$ an V^{-1} exists

7) $|E| = |f(x) - P_1(x)| = \left[\frac{f''(\xi)}{2!}(x-x_0)(x-x_1)\right]$

find max value

let $g(x) = x^2 - (x_0+x_1)x + x_0 x_1 \Rightarrow g'(x) = 2x - (x_0+x_1) = 0$

at extrema

$\Rightarrow$ extrema at $\frac{(x_0+x_1)}{2}$

$\Rightarrow g\left(\frac{x_0+x_1}{2}\right) = \left(\frac{x_1-x_0}{2}\right)\left(\frac{x_0-x_1}{2}\right) = -\frac{(x_1-x_0)^2}{4} = -\frac{h^2}{4}$ since $h = x_1 - x_0$

$\therefore |E| = \left[\frac{f''(\xi)}{2}\left(\frac{-h^2}{4}\right)\right] \leq \frac{h^2}{8} \max_{x \in [x_0, x_2]} |f''(x)|$

8) $f(x) = ae^{bx}$

$\Rightarrow y = ae^{bx} \Rightarrow \ln|y| = \ln(ae^{bx})$

$\Rightarrow \ln(y) = \ln(a) + \ln(e^{bx})$

$\Rightarrow \ln(y) = \ln(a) + bx$

$\Rightarrow \boxed{\ln(y) = A + bx}$ not $\boxed{A = \ln(a) \Rightarrow a = e^{A}}$

Table becomes:

x	1	2	3	4
$\ln(y)$	0	.6932	1.946	3.332

Answer
$f(x) = .2673e^{1.1249x}$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} A \\ b \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ .6932 \\ 1.946 \\ 3.332 \end{bmatrix}}_{b}$$

$\boxed{f(4) \approx 20468}$

$Ax = b \Rightarrow A^TAx = A^Tb \Rightarrow x = (A^TA)^{-1}A^Tb$

calculator drill!

Note: You can also do this using equations on p. 420

$A = -1.3195$, $\boxed{b = 1.1249} \Rightarrow a = e^{-1.3195} = \boxed{.2673 = a}$

9)

$$= f(x+2h) = f(x) + 2h f'(x) + \frac{(2h)^2}{2!} f''(\xi) + \frac{(2h)^3}{3!} f'''(\xi)$$

$$f(x-2h) = f(x) - 2h f'(x) + \frac{(-2h)^2}{2!} f''(\xi) + \frac{(-2h)^3}{3!} f'''(\xi)$$

ERROR TERMS

next term

Total Error is $\frac{1}{4h}\left[\frac{4h^2}{2} f''(\xi) - \frac{4h^2}{2} f''(\xi)\right] = 0$

$\Rightarrow$ NOTE: ERROR CAN'T BE ZERO, THEREFORE PROCEED IN NEXT TERM IN TAYLOR SERIES

$$\Rightarrow \frac{1}{4h}\left[\frac{8h^3}{6} f'''(\xi) - \frac{-8h^3}{6} f'''(\xi)\right]$$

$$= \frac{1}{4h}\left[\frac{16h^3}{6} f'''(\xi)\right] = \frac{4h^2}{6} f'''(\xi) = \boxed{\frac{2}{3} h^2 f'''(\xi)}$$

$$\therefore \quad E = O(h^2)$$

10)

	$O(h^2)$	$O(h^3)$	$O(h^4)$
D_h	.7378169125		
$D_{h/2}$	.7187845413	.7117737509	
$D_{h/4}$	.7104251526	.7076386897	.7070479667

you need this info which wasn't provided

$$\frac{4D_{h/2} - D_h}{3} \qquad \frac{8D_{h/2} - D_h}{7}$$

$$\frac{2^2 D_{h/2} - D_h}{2^2 - 1} \qquad \frac{2^3 D_{h/2} - D_h}{2^3 - 1}$$