Score:

Name: Solutions

Section (circle one): 1 2 3 4 5 6
Team (circle one): a b c d e f

SM365 – Test 2 - Fall 2011

1. Vector Norms: Given the vector $\vec{b} = [\overset{-3,4}{-\cancel{1,1,2}}]^T$, find (do not use norm functions on TI):

a. $\|\vec{b}\|_1$

$= |3| + |-4| = \boxed{7}$

b. $\|\vec{b}\|_2 = \left(3^2 + 4^2\right)^{1/2} = \boxed{5}$

c. $\|\vec{b}\|_5 = \left(3^5 + 4^5\right)^{1/5} = \boxed{4.174}$

d. $\|\vec{b}\|_\infty$

$= \max\{|3|, |-4|\} = \boxed{4}$

No marks on this table		
TH (10 pts)		
1 (10 pts)		
2 (20 pts)		
3 (20 pts)		
4 (20 pts)		
5 (20 pts)		
cumm.		

2. Matrix Norms:

a. Prove that $\|AB\| \le \|A\|\|B\|$ for any $n \times n$ matrices A and B.

① $\|AB\vec{x}\| \overset{?}{=} \|A(B\vec{x})\|$ Associative Prop Matrix Mult.

② $\|A(B\vec{x})\| \le \|A\|\|B\vec{x}\| \le \|A\|\|B\|\|\vec{x}\|$ Consistency Property

③ $\dfrac{\|AB\vec{x}\|}{\|\vec{x}\|} \le \dfrac{\|A\|\|B\|\|\vec{x}\|}{\|\vec{x}\|} = \|A\|\|B\|$

④ Since $\max\limits_{\|\vec{x}\| \neq 0} \dfrac{\|AB\vec{x}\|}{\|\vec{x}\|} = \|AB\| \Rightarrow \|AB\| \le \|A\|\|B\|$ def of matrix norm

b. Given: $A = \begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 5 \\ 7 & 2 \end{bmatrix}$, show numerically that $\|AB\|_\infty \le \|A\|_\infty \|B\|_\infty$ (do not use norm functions on TI).

$AB = \begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -1 & 5 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 17 \\ 37 & 0 \end{bmatrix} \Rightarrow \|AB\|_\infty = \max(21, 37) = 37$

$\|A\|_\infty = \max(4, 7) = 7$

$\|B\|_\infty = \max(6, 9) = 9$

$\Rightarrow 37 \le (7)(9) = 63$ ✓✓

c. Find $\kappa(A)$ based on $\|\cdot\|_\infty$. Use Cramer's Rule to find A^{-1}. If your calculator has a function to determine this directly, DO NOT USE IT!

$\|A\|_\infty = 7$

$A^{-1} = \frac{1}{17}\begin{bmatrix} 5 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5/17 & -1/17 \\ 2/17 & 3/17 \end{bmatrix} \Rightarrow \|A^{-1}\|_\infty = \max\left(6/17, 5/17\right) = 6/17$

$\therefore\ \kappa(A) = (6/17)(7) = \frac{42}{17} \approx 2.47$

Name:______________________________

3. Given the system of equations: $\begin{cases} \cancel{3x + y = -1} \\ \cancel{-2x + 5y = 12} \end{cases}$ $\quad \begin{matrix} +2x - 5y = -12 \\ -3x - y = 1 \end{matrix}$

   a. Solve this system Gaussian elimination with partial pivoting.

Pivot on -3

$$\begin{matrix} 2x - 5y = -12 \\ -3x - y = 1 \end{matrix} \xrightarrow{R1 + \frac{2}{3}R_2} \begin{cases} -\frac{17}{3}y = -\frac{34}{3} \Rightarrow \\ -3x - 2 = 1 \Rightarrow \end{cases} \boxed{\begin{matrix} y = 2 \\ x = -1 \end{matrix}}$$

   b. Solve this system using LU factorization with no pivoting.

Factor

$$\begin{matrix} 2x - 5y = -12 \\ -3x - y = 1 \end{matrix} \Rightarrow R3 + \tfrac{3}{2}R_1 \quad \begin{bmatrix} 2 & -5 \\ (-3/2) & -17/2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ (-3/2) & 1 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ 0 & -17/2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -5 \\ -3 & -1 \end{bmatrix} \checkmark$$

Solve

$$\underbrace{LU\vec{x}}_{\vec{z}} = \vec{b} \Rightarrow L\vec{z} = \vec{b} \Rightarrow \begin{bmatrix} 1 & 0 \\ -3/2 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -12 \\ 1 \end{bmatrix}$$

$\Rightarrow z_1 = -12$

$-3/2\, z_1 + z_2 = -1 \Rightarrow z_2 = -1 + \frac{3}{2}z_1 = -1 + \frac{3}{2}(-12) = -17$

$$U\vec{x} = \vec{z} \Rightarrow \begin{bmatrix} 2 & -5 \\ 0 & -17/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -12 \\ -17 \end{bmatrix} \Rightarrow$$

$2x_1 - 5x_2 = -12$

$-17/2\, x_2 = -17 \Rightarrow \boxed{x_2 = 2}$

$2x_1 - 5(2) = -12 \Rightarrow 2x_1 = -2$

$\Rightarrow \boxed{x_1 = -2}$

4. Given the system of equations: $\begin{cases} 3x + \ \ y = -1 \\ -2x + 5y = \ \ 12 \end{cases}$.

a. Set up a Jacobi fixed point iteration scheme $\vec{x}^{(k+1)} = T\vec{x}^{(k)} + \vec{c}$.

$$x^{(k+1)} = \frac{1}{3}(-1-y^{(k)}) = -\frac{1}{3} - \frac{1}{3}y^{(k)}$$

$$y^{(k+1)} = \frac{1}{5}(12+2x^{(k)}) = \frac{12}{5} + \frac{2}{5}x^{(k)}$$

$$\Rightarrow \boxed{\vec{x}^{\,k+1} = \begin{bmatrix} 0 & -1/3 \\ 2/5 & 0 \end{bmatrix}\vec{x}^{(k)} + \begin{bmatrix} -1/3 \\ 12/5 \end{bmatrix}}$$

b. Based on matrix T, will this scheme converge to a solution? Why? Do not ~~You may~~ use TI ~~as~~ to ~~necessary to~~ find eigenvalues.

find $\rho(T)$

$$\det\begin{pmatrix} -\lambda & -1/3 \\ 2/5 & -\lambda \end{pmatrix} = \lambda^2 + \frac{2}{15} = 0 \Rightarrow \lambda^2 = \frac{-2}{15}$$

$$\Rightarrow \lambda = \pm\sqrt{\frac{2}{15}}\,i \Rightarrow |\lambda| = \sqrt{\frac{2}{15}} < 1$$

since $\rho(T) < 1$, scheme will converge

c. Perform the 1st two steps of the Jacobi iteration scheme of $\vec{x}^{(0)} = \{0, 0, 0\}^T$

$$\begin{cases} x^{(1)} = -\frac{1}{3} - \frac{1}{3}(0) = -1/3 \\ y^{(1)} = \frac{12}{5} + \frac{2}{5}(0) = 12/5 \end{cases}$$

$$x^{(2)} = -\frac{1}{3} - \frac{1}{3}\left(\frac{12}{5}\right) = -\frac{1}{3} - \frac{4}{5} = -\frac{5-12}{15} = \boxed{-\frac{17}{15} \approx -1.133}$$

$$y^2 = \frac{12}{5} + \frac{2}{5}\left(-\frac{1}{3}\right) = \frac{12}{5} - \frac{2}{15} = \frac{36-2}{15} = \boxed{\frac{34}{15} \approx 2.267}$$

Name:______________________________

5. Given the system of equations: $\begin{cases} 3x + \ \ y = -1 \\ -2x + 5y = \ \ 12 \end{cases}$.

   a. Set up a Gauss-Seidel fixed point iteration scheme $\vec{x}^{(k+1)} = T\vec{x}^{(k)} + \vec{c}$.

As Before: $\quad x^{(k+1)} = -\frac{1}{3} - \frac{1}{3}y^{(k)}$

But $\quad y^{(k+1)} = \frac{12}{5} + \frac{2}{5}x^{(k+1)} = \frac{12}{5} + \frac{2}{5}\left(-\frac{1}{3} - \frac{1}{3}y^{(k)}\right)$

$$= \frac{12}{5} - \frac{2}{15} - \frac{2}{15}y^{(k)} =$$

$$= \frac{34}{15} - \frac{2}{15}y^{(k)}$$

$$\boxed{\vec{x}^{(k+1)} = \begin{bmatrix} 0 & -1/3 \\ 0 & -2/15 \end{bmatrix} \vec{x} + \begin{bmatrix} -1/3 \\ 34/15 \end{bmatrix}}$$

   b. Perform the 1st two steps of the Gauss-Seidel iteration scheme of $\vec{x}^{(0)} =$ $[1,1]^T$

$$\begin{cases} x^{(1)} = -\frac{1}{3} - \frac{1}{3}(1) = -\frac{2}{3} \approx -.6667 \\ y^{(1)} = \frac{34}{15} - \frac{2}{15}(1) = \frac{32}{15} \approx 2.1333 \end{cases}$$

$$x^{(2)} = -\frac{1}{3} - \frac{1}{3}\left(\frac{32}{15}\right) = \boxed{-\frac{47}{45} \approx -1.0444}$$

$$y^{(2)} = \frac{34}{15} - \frac{2}{15}\left(\frac{32}{15}\right) = \boxed{\frac{446}{225} \approx 1.9822}$$