Score:

Name:______________________________

Section (circle one): 1 2 3 4 5 6
Team (circle one): a b c d e f

SM365 – Test 1 - Fall 2011

1. Derive a 3-term Taylor series for the $\ln(x)$ centered on $x = 1$. What is the theoretical error bound if we restrict the approximation to the interval $.8 \le x \le 1.2$?

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$f^{(n)}(1)/n!$
0	$\ln(x)$	0	0
1	$1/x$	1	1
2	$-1/x^2$	-1	$-1/2$
3	$+2/x^3$	2	$1/3$
4	$-6/x^4$	-6	$-1/4$

$$\boxed{P_3 = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3}$$

$$|R_4| = \left|\frac{-6}{\xi^4}\frac{(x-1)^4}{4!}\right| = \left|+\frac{1}{4\xi}(x-1)^4\right|$$

let $\xi = .8$ $\quad x = 1.2$ ← maxes out R_N

$$|R_4| \le \left|\frac{1}{4(.8)^4}(.2)^4\right| =$$

$$\Rightarrow \boxed{R_4 \le .000977}$$

No marks on this table		
ST (10 pts)		
1 (15pts)		
2 (10 pts)		
3 (15 pts)		
4 (15 pts)		
5 (10 pts)		
6 (10 pts)		
7 (15 pts)		
cumm.		

2. A remote civilization was discovered to do all of its numerical calculations in a the floating point number system F(10,1,0,2). Numbers are truncated by rounding. Answer the following questions:

   a. List the 27 numbers that make up this system.

   {.1, .2, .3, .4, .5, .6, .7, .8, .9, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 30, 40, 50, 60, 70, 80, 90}

   b. Let a=~~x~~.25, b=.39, and c=.82. Assume you are restricted to working in F(10,1,0,2). Calculate $(a + b) + c$. What is the roundoff error?

   $a+b = Fl(Fl(a) + Fl(b)) = Fl(.3 + .4) = .7$

   $(a+b)+c = Fl(.7 + Fl(c)) = Fl(.7 + .8) = 1.5 = 2$

   a+b+c would equal = .25 + .39 + .82 = 1.46 "normally"

   ∴ roundoff error +.54

   c. Calculate $a + (b + c)$. What is the roundoff error?

   $(b+c) = .4 + .8 = 1.2 = 1$

   $a + 1. = .3 + 1 = 1.3 = \boxed{1}$

   ∴ round off error +.46

   d. What conclusion can you make about the laws of arithmetic from the above calculations?

   The associative property does not apply to addition

   e. What is the machine precision for the system described above?

   By definition:

   $u = \frac{1}{2} B^{1-s} = \frac{1}{2}(10)^{1-1} = \frac{1}{2}$ or .5

Name:______________________________

3. Given the sequence $p_n = \frac{3^n+1}{3^n-1}$.   NOTE $\lim_{n\to\infty} \left|\frac{3^n+1}{3^n-1}\right| = 1$.

   a. Determine the rate of convergence.

$$\left|\frac{3^n+1}{3^n-1} - 1\right| = \left|\frac{\cancel{3^n}+1-\cancel{3^n}+1}{3^n-1}\right| = \left|\frac{2}{3^n-1}\right| \doteq 2\left|\frac{1}{3^n-1}\right|$$

Rate of convergence is $O\left(\frac{1}{3^n}\right)$

   b. The sequence was used to create the following table of data. From this table what can you conclude about the *order of convergence* and the *asymptotic error*

$\alpha = 1$

n	p	$p_n - 1$	$\frac{\lvert p_{n+1}-1\rvert}{\lvert p_n - 1\rvert}$	$\frac{\lvert p_{n+1}-1\rvert}{\lvert p_n - 1\rvert^2}$	$\frac{\lvert p_{n+1}-1\rvert}{\lvert p_n - 1\rvert^3}$
0	0	-1			
1	0.5	-0.5	0.5	0.5	0.5
2	0.8	-0.2	0.4	0.8	1.6
3	0.928571	0.07143	0.357143	1.785714	8.928571
4	0.97561	0.02439	0.341463	4.780488	66.92683
5	0.991803	-0.0082	0.336066	13.77869	564.9262
6	0.99726	0.00274	0.334247	40.77808	4974.926
7	0.999086	0.00091	0.333638	121.7779	44448.93

↑ approaching 1/3

$$\boxed{\begin{array}{l}\alpha = 1 \\ \lambda = 1/3\end{array}}$$

   c. Analytically derive the *order of convergence* and the *asymptotic error* for the sequence in #3.

$$\lim_{n\to\infty} \frac{\left|\frac{3^{n+1}+1}{3^{n+1}-1} - 1\right|}{\left|\frac{3^n+1}{3^n-1} - 1\right|} = \lim_{n\to\infty} \frac{\left|\frac{\cancel{3^{n+1}}+1-\cancel{3^{n+1}}+1}{3^{n+1}-1}\right|}{\left|\frac{3^n+1-3^n+1}{3^n-1}\right|} = \lim_{n\to\infty} \left|\frac{\frac{2}{3^{n+1}-1}}{\frac{2}{3^n-1}}\right|$$

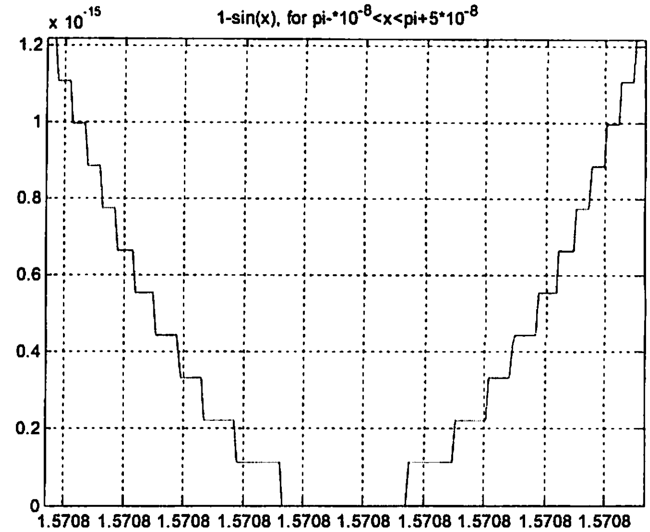
$$= \lim_{n\to\infty} \left|\frac{3^n \cancel{-1}}{3^{n+1}\cancel{-1}}\right| = \lim_{n\to\infty} \left|\frac{1}{3}\right| = 1/3$$

4. The function $y = 1 - \sin(x)$ was plotted on MATLAB on the interval $|x - \pi| \leq 5 * 10^{-8}$. Answer the following:

   a. What is anomalous about this graph?

   b. To what do you attribute this anomaly?

   c. Modify the function to correct the anomaly.



a) It should be "smooth" not stair stepped

b) We are subtract 2 nearly equal small numbers. There is significant round off error

c) eliminate the subtraction step

$$1-\sin(x)\left(\frac{1+\sin(x)}{1+\sin(x)}\right) = \frac{1-\sin^2(x)}{1+\sin(x)}$$

$$\Rightarrow \sin^2 x + \cos^2(x) = 1 \Rightarrow \sin^2(x) = 1 - \cos^2(x)$$

$$\Rightarrow \frac{1-(1-\cos^2(x))}{1+\sin(x)} = \boxed{\frac{\cos^2(x)}{1+\sin(x)}}$$

6. Recast Problem 5 as a fixed point problem.
   a. Perform 4 iterations starting with $x_0 = .2$.

$x - \sin(\sqrt{x}) = 0 \Rightarrow \quad x = \sin\sqrt{x}$

$x_0 = .2$

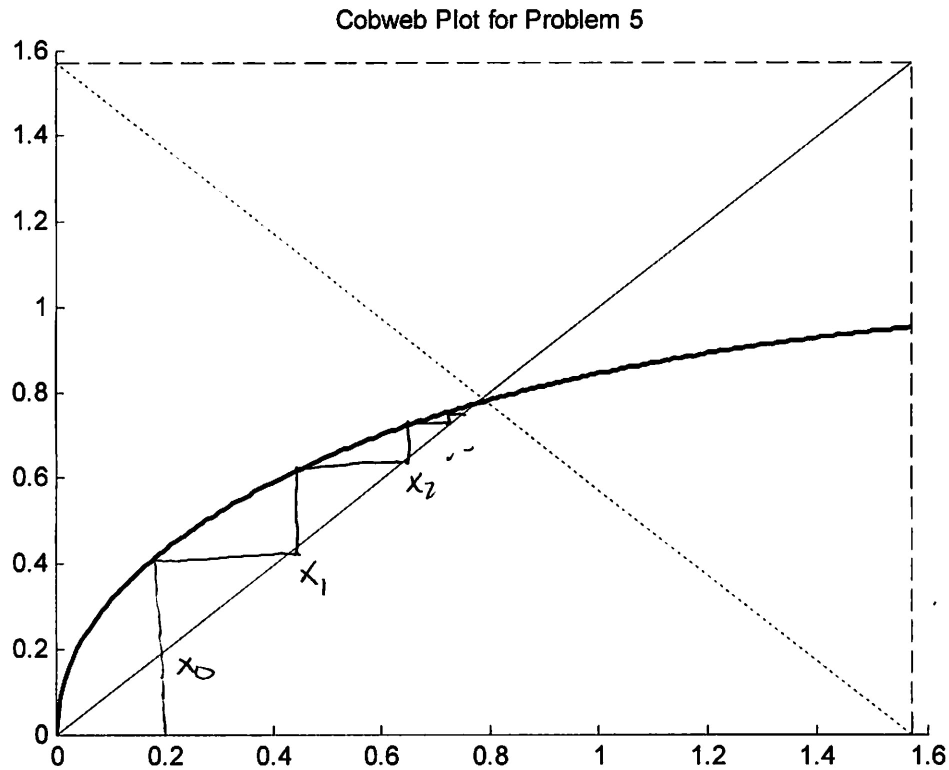
$x_1 = \sin(\sqrt{.2}) = .4325$

$x_2 = .6112$

$x_3 = .7046$

$\boxed{x_4 = .7412}$

   b. Carefully sketch the dynamic of the 4 iterations on the plot below.

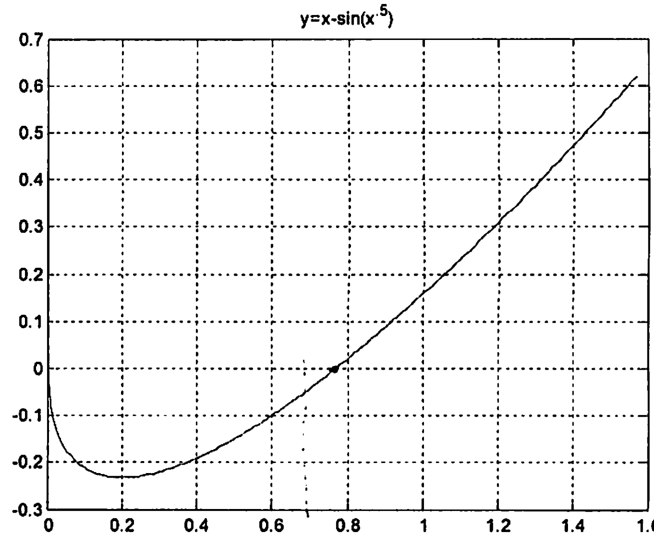


   c. Referencing fixed point theorems, does the plot above indicate that conditions are met to guarantee convergence?

No the slope near 0 is greater than 1!

Name:____________________________

5. Use the bisection method to estimate the 1st root of $f(x) = x - \sin(\sqrt{x})$. Stop after 4 iterations.



$a = .6 \qquad b = .8$

$f(a) < 0 \qquad f(b) > 0$

① $\frac{a+b}{2} = .7 \Rightarrow f(.7) < 0$

$\Rightarrow a = .7 \quad b = .8$

② $\frac{a+b}{2} = .75 \Rightarrow f(.75) = -.0118 < 0$

$\therefore a = .75 \quad b = .8$

③ $\frac{a+b}{2} = .7750 \quad f(.7750) = .004 > 0$

$\therefore a = .75 \quad b = .7750$

④ $\frac{a+b}{2} = .7625$ STOP

Name:______________________________

7. How would you set up problem 5 so that you could find the root using Newton's method, i.e. find $g(x)$. Do not perform any iterations.

$$f(x) = x - \sin(x^{1/2})$$

$$f'(x) = 1 - \tfrac{1}{2}x^{-1/2}\cos(x^{1/2})$$

$$\boxed{g(x) = x - \frac{x - \sin(x^{1/2})}{1 - \frac{1}{2x^{1/2}}\cos(x^{1/2})}}$$

$$\boxed{g(x) = x - \frac{2x^{3/2} - 2x^{1/2}\sin(x^{1/2})}{2x^{1/2} - \cos(x^{1/2})}}$$

↑ or if you prefer