NAME: Solutions

Each problem is worth 10 points

1. Determine the value of the expression:

$$\frac{e}{7} + \sqrt{2}\ln\pi$$

using 4-digit rounding arithmetic. Compute the absolute and the relative error.

4 digit arith

$e = 2.718$

$\pi = 3.142$

$\sqrt{2} = 1.414$

① $\ln(\pi) = \ln(3.142) = 1.145$

② $\sqrt{2}\ln(\pi) = 1.414 \times 1.145 = 1.619$

③ $\frac{e}{7} = \frac{2.718}{7} = .3883$

$\Rightarrow \frac{e}{7} + \sqrt{2}\ln(\pi) = .3883 + 1.619 = \boxed{2.007}$

TI 200 $\left\{ \frac{e}{7} + \sqrt{2}\ln(\pi) = 2.007218505 \right.$

abs error $= |2.007 - 2.007218505| = \boxed{2.185053 \times 10^{-4}}$

rel error $= \frac{\text{abs err}}{2.007218505} = \boxed{1.088598 \times 10^{-4}}$

2. Compute the following limits and determine the corresponding rate of convergence:

(a) $\lim_{n\to\infty} \frac{\sin n}{n}$ $\Rightarrow$ $-\frac{1}{n} \le \frac{\sin(n)}{n} \le \frac{1}{n}$

$\Rightarrow \lim_{n\to\infty} -\frac{1}{n} \le \lim_{n\to\infty} \frac{\sin(n)}{n} \le \lim_{n\to\infty} \frac{1}{n}$

$\Rightarrow 0 \le \lim_{n\to\infty} \frac{\sin(n)}{n} \le 0 \Rightarrow \boxed{\lim_{n\to\infty} \frac{\sin(n)}{n} = 0}$

$\left|\frac{\sin(n)}{n} - 0\right| \le \left|\frac{1}{n}\right| \Rightarrow$ $\boxed{\text{Rate of convergence } O(\tfrac{1}{n})}$

(b) $\lim_{x\to 0} \frac{e^x - \cos x - x}{x^2}$ $= \lim_{x\to 0} \frac{e^x + \sin(x)}{2x} = \lim_{x\to 0} \frac{e^x + \cos(x)}{2} = \frac{1+1}{2} = \boxed{1}$

$\left|\frac{e^x - \cos x - x}{x^2} - 1\right|$ Tay Ser $\begin{cases} e^x \approx 1 + x + \frac{x^2}{2} \ldots \\ \cos(x) \approx 1 - \frac{x^2}{2} + \frac{x^4}{4!} \ldots \end{cases}$

↑ use 1st 3 term

$\left|\frac{\cancel{1} + \cancel{x} + \frac{x^2}{2} - \cancel{1} + \frac{x^2}{2} - \frac{x^4}{4!} - \cancel{x}}{x^2} - 1\right|$

$= \left|\frac{x^2 - x^4/4!}{x^2} - 1\right| = \left|1 - \frac{1}{4!}x^2 - 1\right| = \left|\frac{1}{4!}x^2\right|$

$\Rightarrow \boxed{ROC = O(x^2)}$

3. (a) Verify that the function

$$f(x) = e^{3x} - x^2 - 3, \qquad (0, 1)$$

has a zero on the indicated interval.

$f(0) = 1 - 0 - 3 = -2 < 0$

$f(1) = e^3 - 1 - 3 \approx 16.09 > 0$

Therefore by intermediate value Theorem there lies at least on root inbetween!

(b) Perform the first four (4) iterations of the bisection method and verify that each approximation satisfies the theoretical error bound of the bisection method. The exact location of the zero is 0.382039055228. **Show all your work!**

	Error $\lvert p - p_n \rvert$	<	Error $(b-a)/2^n$	
$n=1$ $f(.5) = 1.23 > 0$	.11796	<	.5	✓
$(0+.5)/2$				
$n=2$ $f(.25) = -.945 < 0$	.13204	<	.25	✓
$(.25+.5)/2$				
$n=3$ $f(.375) = -.060 < 0$	.00704	<	.125	✓
$(.375+.5)/2$				
$n=4$ $f(.4375) = .524 > 0$	.05546	<	.0625	✓

$$E = \frac{b-a}{2^n}$$

4. For the augmented matrix

$$\left[\begin{array}{ccc|c} 3 & 4 & 6 & -4 \\ 4 & 1 & 4 & 9 \\ 2 & 3 & 1 & 0 \end{array}\right] \begin{array}{l} \Rightarrow R_2 - \frac{4}{3}R_1 \\ \Rightarrow R_3 = \frac{2}{3}R_1 \end{array}$$

show the matrix that results after reduction of the first column using

(a) Gauss elimination with no pivoting

PIVOT $\Rightarrow$ 3

$R_2 - \frac{4}{3} R_1$

$R_3 - \frac{2}{3} R_1$

$$\Rightarrow \boxed{\begin{bmatrix} 3 & 4 & 6 & -4 \\ 0 & -13/3 & -4 & 43/3 \\ 0 & 1/3 & -3 & 8/3 \end{bmatrix}}$$

(b) Gauss elimination with partial pivoting.

Pivot $\Rightarrow$ 4

$R_1 - \frac{3}{4} R_2$

$R_3 - \frac{1}{2} R_2$

$$\boxed{\begin{bmatrix} 0 & 13/4 & 3 & -43/4 \\ 4 & 1 & 4 & 9 \\ 0 & 5/2 & -1 & -9/2 \end{bmatrix}}$$

OK if you did a row swap 1st

5. Consider the linear system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -2 & 4 & 0 \\ -1 & 2 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

Approximate, starting with zero initial vector, the solution $\mathbf{x}$ by carrying out 2 iterations:

(a) Jacobi iteration

$$3x + y + z = 4 \Rightarrow x = \tfrac{1}{3}(4 - y - z)$$
$$-2x + 4y = 1 \Rightarrow y = \tfrac{1}{4}(1 + 2x)$$
$$-x + 2y + 6z = 2 \Rightarrow z = \tfrac{1}{6}(2 + x - 2y)$$

$x_0 = 0$ $\quad x_1 = \frac{4}{3}$ $\quad x_2 = \frac{1}{3}\left(4 - \frac{1}{4} - \frac{1}{3}\right) = \boxed{41/36 \approx 1.139}$

$y_0 = 0$ $\quad y_1 = \frac{1}{4}$ $\quad y_2 = \frac{1}{4}\left(1 + 2\left(\frac{4}{3}\right)\right) = \boxed{11/12 \approx .917}$

$z_0 = 0$ $\quad z_1 = \frac{1}{3}$ $\quad z_2 = \frac{1}{6}\left(2 + \frac{4}{3} - 2\left(\frac{1}{4}\right)\right) = \boxed{17/36 \approx .472}$

(b) Gauss-Seidel iteration

$\boxed{x_1 = 4/3 \approx 1.33}$

$y_1 = \frac{1}{4}\left(1 + 2\left(\frac{4}{3}\right)\right) = \boxed{\frac{11}{12} \approx .917}$

$z_1 = \frac{1}{6}\left(2 + \frac{4}{3} - 2\left(\frac{11}{12}\right)\right) = \boxed{1/4 = .25}$

$\boxed{\begin{matrix} x_1 \approx 1.33 \\ y_1 \approx .917 \\ z_1 \approx .25 \end{matrix}}$

$x_2 = \frac{1}{3}\left(4 - \frac{11}{12} - \frac{1}{4}\right) = \boxed{\frac{17}{18} \approx .944}$

$y_2 = \frac{1}{4}\left(1 + 2\left(\frac{17}{18}\right)\right) = \boxed{13/18 \approx .722}$

$\boxed{\begin{matrix} x_2 \approx .944 \\ y_2 \approx .722 \\ z_2 \approx .25 \end{matrix}}$

$z_2 = \frac{1}{6}\left(2 + \frac{17}{18} - 2\left(\frac{13}{18}\right)\right) = \boxed{\frac{1}{4} \approx .25}$

6. Prove the following Theorem:
Let A be a non-singular matrix, $\tilde{\mathbf{x}}$ be an approximate solution to the linear system $A\mathbf{x} = \mathbf{b}$, $\mathbf{r} = A\tilde{\mathbf{x}} - \mathbf{b}$ and $\mathbf{e} = \tilde{\mathbf{x}} - \mathbf{x}$. Then, for any natural matrix norm $\|\cdot\|$,

$$\frac{1}{\|A\|}\|\mathbf{r}\| \le \|\mathbf{e}\| \le \|A^{-1}\|\|\mathbf{r}\|$$

① $\vec{r} = A\tilde{\vec{x}} - \vec{b} \rightarrow \vec{r} = A\tilde{\vec{x}} - A\vec{x}$ → using eqn's given above

$\rightarrow r = A(\tilde{\vec{x}} - \vec{x})$ → factor out A

$\rightarrow r = A\vec{e}$ → definition of $\vec{e}$

$\Rightarrow \vec{e} = A^{-1}\vec{r}$ property of matrix inverse and since A is non-singular

② $\|\vec{e}\| = \|A^{-1}\vec{r}\| \le \|A^{-1}\|\|r\|$ → Consistency property

③ $\|\vec{e}\| \le \|A^{-1}\|\|r\|$

④ since $\vec{r} = A\vec{e}$ → shown above

then $\|\vec{r}\| = \|A\vec{e}\| \le \|A\|\|\vec{e}\|$ → consistency property

$\therefore \|\vec{r}\| \le \|A\|\|\vec{e}\| \rightarrow \frac{\|\vec{r}\|}{\|A\|} \le \|\vec{e}\|$

⑤ $\therefore \frac{\|\vec{r}\|}{\|A\|} \le \|\vec{e}\| \le \|A^{-1}\|\|\vec{r}\|$ → putting 3 & 4 together

$\boxed{\text{QED}}$

7. Let $\kappa(A)$ be the condition number of a matrix A.

(a) Show $\kappa(A) \geq 1$ when a natural matrix norm is used.

① Use Theorem from problem 6 which implies

$$\frac{1}{\|A\|}\|\vec{r}\| \leq \|A^{-1}\|\|\vec{r}\|$$

$$\Rightarrow \frac{1}{\|A\|} \leq \|A^{-1}\| \Rightarrow 1 \leq \underbrace{\|A\|\|A^{-1}\|}_{k(A)} \Rightarrow 1 \leq k(A)$$

② $\therefore\ k(A) \geq 1$ $\boxed{\text{QED}}$

Better Proof:

① $k(A) = \|A\|\|A^{-1}\|$

② $\|A\|\|A^{-1}\| \geq \|A A^{-1}\| = \|I\| = 1$

consistency

③ $k(A) \geq 1$ QED

(b) Find $\kappa_\infty(A)$ for $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

$$k(A) = \|A\|_\infty \|A^{-1}\|_\infty$$

$$\|A\|_\infty = \max[(1+2),(4+3)] = 7$$

$$A^{-1} = \frac{1}{3-8}\begin{vmatrix} 3 & -2 \\ -4 & 1 \end{vmatrix} = -\frac{1}{5}\begin{vmatrix} 3 & -2 \\ -4 & 1 \end{vmatrix} = \begin{vmatrix} -3/5 & 2/5 \\ 4/5 & -1/5 \end{vmatrix}$$

$$\|A^{-1}\|_\infty = \max[(3/5+2/5),(4/5+1/5)] = 1$$

$$\Rightarrow \boxed{k(A) = 7}$$

8. Using Taylor series, derive the error term for the approximation

$$f'''(x) \approx \frac{1}{2h^3}[f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)].$$

$$f(x+2h) = f(x) + \frac{2h}{1!}f'(x) + \frac{4h^2}{2!}f''(x) + \frac{8h^3}{3!}f'''(x) + \frac{16h^4}{4!}f^{(4)}(\xi) + \frac{32h^5}{5!}f^{(5)}(\xi)$$

$$f(x+h) = \ldots \quad \frac{h^4}{4!}f^{(4)}(\xi) + \frac{h^5}{5!}f^{(5)}(\xi)$$

$$f(x-h) = \ldots \quad \frac{h^4}{4!}f^{(4)}(\xi) - \frac{h^5}{5!}f^{(5)}(\xi)$$

$$f(x-2h) = \ldots \quad \frac{16h^4}{4!}f^{(4)}(\xi) - \frac{32h^5}{5!}f^{(5)}(\xi)$$

$$E_T = \frac{1}{2h^3}\left[(16h^4 - 2h^4 + 2h^4 - 16h^4)\frac{f^{(4)}(\xi)}{4!}\right] = 0$$

move to next error term

$$E_T = \frac{1}{2h^3}\left[(32 - 2 - 2 + 32)\frac{h^5}{5!}f^{(5)}(\xi)\right)$$

$$= \frac{1}{2h^3}\left(\frac{60}{120}h^5 f^{(5)}(\xi)\right) = \boxed{\frac{1}{4}h^2 f^{(5)}(\xi)}$$

9. Show that if $u(x)$ is any function that interpolates $f(x)$ at $x_0, x_1, \ldots, x_{n-1}$, and if $v(x)$ is a function that interpolates $f(x)$ at $x_1, x_2, \ldots, x_n$, then the function

$$w(x) = \frac{(x_n - x)u(x) + (x - x_0)v(x)}{x_n - x_0}$$

interpolates $f(x)$ at $x_0, x_1, \ldots, x_n$.

① note for $u(x)$, $u(x_i) = f(x_i)$ for $i = 0, n-1$

② " for $v(x)$, $v(x_i) = f(x_i)$ for $i = 1, n$

③ let $x = x_0$

$$w(x_0) = \frac{(x_n - x_0)u(x_0) + \overbrace{(x_0 - x_0)}^{0}v(x_0)}{x_n - x_0} = u(x_0) = f(x_0)$$ ✓

④ let $x_i = x_1, \ldots x_{n-1}$

$$w(x_i) = (x_n - x_i)\overbrace{u(x_i)}^{f(x_i)} + (x_i - x_0)\overbrace{v(x_i)}^{f(x_i)}$$

$$= \frac{(x_n - x_i + x_i - x_0) f(x_i)}{(x_n - x_0)} = f(x_i)$$ ✓✓

⑤ let $x = x_n$

$$w(x_n) = \frac{\overbrace{(x_n - x_n)}^{0}u(x_n) + (x_n - x_0)v(x_n)}{(x_n - x_0)} = v(x_n) = f(x_n)$$ ✓✓

⑥ $\therefore$ $w(x_i) = f(x_i)$ for $x = 0, \ldots n$

$\therefore$ $w(x)$ is an interpolating polynomial for the data

QED

10. (a) You are given the following table:

x	-1	0	1	2	3
y	2	1	2	-7	10

Find a polynomial, using a divided differences table, that assumes the values in the table.

-1	2				
		-1			
0	1		1		
		1		-2	
1	2		-5		2
		-9		6	
2	-7		13		
		17			
3	10				

$f(x) = 2 - (x+1) + 1(x+1)(x) - 2(x+1)(x)(x-1) + 2(x+1)(x)(x-1)(x-2)$

$\Rightarrow \boxed{f(x) = 2x^4 - 6x^3 - x^2 + 6x + 1}$

(b) Use the method of least squares to find a logarithmic fitting $f(x) = a + b\ln x$ for the data:

x	2.0	2.5	3.0	3.5	4.0
y	11.73	14.54	16.84	18.78	20.46

$$\begin{array}{cc} a & b(\ln(x)) \\ \end{array}$$
$$\begin{bmatrix} 1 & \ln(2) \\ 1 & \ln(2.5) \\ 1 & \ln(3) \\ 1 & \ln(3.5) \\ 1 & \ln(4) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 11.73 \\ 14.54 \\ 16.84 \\ 18.78 \\ 20.46 \end{bmatrix}$$

$A\vec{x} = \vec{b}$

$A^TA\vec{x} = A^T\vec{b}$

$\vec{x} = (A^TA)^{-1}A^T\vec{b}$

TI work

$a \approx 3$
$b \approx 12.6$

$\boxed{f(x) \simeq 3 + 12.6\ln x}$