SM315 Lecture Notes
Finite Difference Approximations of Derivatives
Homework: (228) 4,5

x	f(x)
0	1
0.1	1.092
0.2	1.176
0.3	1.264
0.4	1.368
0.5	1.5
0.6	1.672
0.7	1.896
0.8	2.184
0.9	2.548
1	3

1. Consider Table at Left: Use Data to Estimate 1st Derivative at x=.5

a. Recall Taylor Series:

$$f(x+\Delta x) \approx f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + \frac{\Delta x^3}{3!} f'''(x) + \ldots \frac{\Delta x^n}{n!} f^{(n)}(x) + O(\Delta x^n)$$

b. Use Taylor series to get a finite difference approximation of the 1st derivative. To do this I need to look at least one point forward or one point backward on my table.

c. <u>Forward Difference Approximation for 1st Derivative</u>:

- Consider: $\begin{cases} f(x) = f(x) \\ f(x+\Delta x) = f(x) + \Delta x f'(x) + O(\Delta x) \end{cases}$
- Combine this equation in such a way that $f(x)$ is eliminated, i.e.

$$\begin{cases} Af(x) = Af(x) \\ Bf(x+\Delta x) \approx Bf(x) + B\Delta x f'(x) \end{cases} \rightarrow \begin{cases} A+B=0 \\ B\Delta x = 1 \end{cases} \rightarrow \begin{cases} A=-B \\ B = 1/\Delta x \rightarrow A = -1/\Delta x \end{cases}$$

- Therefore $f'(x) \approx -1/\Delta x\, f(x) + 1/\Delta x\, f(x+\Delta x) \rightarrow f'(x) \rightarrow$

$$\boxed{f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}}$$

- Hence $f(1.5) \approx \frac{f(.6) - f(.5)}{.1} = \frac{1.672 - 1.5}{.1} = 1.72$

d. <u>Backward Difference Approximation 1st Derivative</u>:

- Is easily obtained from forward difference formula by letting $x = x - \Delta x$
- Thus: $\boxed{f'(x) \approx \frac{f(x) - f(x-\Delta x)}{\Delta x}}$
- And: $f(1.5) \approx \frac{f(.5) - f(.4)}{.1} = \frac{1.5 - 1.368}{.1} = 1.32$

e. Three-Point Bachward Difference Approximation for 1st Derivative:

- Consider: $$\begin{cases} f(x) = f(x) \\ f(x-\Delta x) = f(x) - \Delta x f'(x) + \dfrac{(\Delta x)^2}{2} f''(x) + O\left((\Delta x)^2\right) \\ f(x-2\Delta x) = f(x) - 2\Delta x f'(x) + \dfrac{(2\Delta x)^2}{2} f''(x) + O\left((\Delta x)^2\right) \rightarrow \\ f(x-2\Delta x) = f(x) - 2\Delta x f'(x) + 2(\Delta x)^2 f''(x) + O\left((\Delta x)^2\right) \end{cases}$$
- Combine this equation in such a way that $f(x)$ is eliminated, i.e.

$$\begin{cases} Af(x) = Af(x) \\ Bf(x-\Delta x) = Bf(x) - B\Delta x f'(x) + B\dfrac{(\Delta x)^2}{2} f''(x) \\ Cf(x-2\Delta x) = Cf(x) - 2C\Delta x f'(x) + 2C(\Delta x)^2 f''(x) \end{cases} \rightarrow \begin{cases} A + B + C = 0 \\ -B\Delta x - 2C\Delta x = 1 \\ \dfrac{B}{2}(\Delta x)^2 + 2C(\Delta x)^2 = 0 \end{cases}$$

$$\rightarrow B = -4C \rightarrow C = \frac{1}{2\Delta x} \rightarrow B = -\frac{2}{\Delta x} \rightarrow A = \frac{3}{2\Delta x}$$

- Therefore $$\boxed{f'(x) \approx \frac{3f(x) - 4f(x-\Delta x) + f(x-2\Delta x)}{2\Delta x}}$$
- Hence $f(1.5) \approx \dfrac{3f(.5) - 4f(.4) + f(.3)}{2(.1)} = 1.46$

f.

Central Difference Approximation 1st Derivative:

- Consider: $\begin{cases} f(x-\Delta x) = f(x) - \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + O(\Delta x^2) \\ f(x) = f(x) \\ f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + O(\Delta x^2) \end{cases}$
- Note: I'll need to go out a 3rd term in the Taylor series to accomplish this analysis. We'll see why in a moment.
- Combine this equation in such a way that $f(x)$ and $f''(x)$ is eliminated,

  i.e. $\begin{cases} Af(x-\Delta x) \approx Af(x) - A\Delta x f'(x) + A\frac{\Delta x^2}{2} f''(x) \\ Bf(x) = Bf(x) \\ Cf(x+\Delta x) \approx Cf(x) + C\Delta x f'(x) + C\frac{\Delta x^2}{2} f''(x) \end{cases}$
- Therefore: $\begin{cases} A+B+C=0 \\ -A\Delta x + C\Delta x = 1 \\ A\frac{\Delta x^2}{2} + C\frac{\Delta x^2}{2} = 0 \end{cases} \rightarrow \begin{cases} A=-C \\ 2C = 1/\Delta x \end{cases} \rightarrow \begin{cases} C = 1/2\Delta x \\ A = -1/2\Delta x \\ B = 0 \end{cases}$
- Thus:

  $f'(x) \approx f(x) + \frac{1}{2\Delta x} f(x+\Delta x) - \frac{1}{2\Delta x} f(x-\Delta x) \rightarrow$

  $$\boxed{f'(x) \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}}$$
- And: $f(1.5) \approx \frac{f(.6) - f(.4)}{.2} = \frac{1.672 - 1.368}{.2} = 1.52$

g. Summary of Three Schemes 1st Derivative:

x	f(x)	f'(x)	FD	FD Error	BD	BD Error	3PT BD	3PT BD Error	CD	CD Error
0.5	1.5	1.5	1.62	0.12	1.32	-0.18	1.48	-.02	1.52	0.02

h.

Central Difference Approximation 2^{nd} Derivative:

- There was enough information in above equations to Approximate 2^{nd} Derivative, i.e. start with:

$$\begin{cases} Af\left(x-\Delta x\right) \approx Af\left(x\right) - A\Delta x f'\left(x\right) + A\dfrac{\Delta x^2}{2} f''\left(x\right) \\ Bf\left(x\right) = Bf\left(x\right) \\ Cf\left(x+\Delta x\right) \approx Cf\left(x\right) + C\Delta x f'\left(x\right) + C\dfrac{\Delta x^2}{2} f''\left(x\right) \end{cases}$$

- This time we wish to eliminate $f\left(x\right)$ and $f'\left(x\right)$:

$$\begin{cases} A+B+C=0 \\ -A\Delta x + C\Delta x = 0 \\ A\dfrac{\Delta x^2}{2} + C\dfrac{\Delta x^2}{2} = 1 \end{cases} \rightarrow \begin{cases} A = C \\ C = 1/\Delta x^2 \end{cases} \rightarrow \begin{cases} C = 1/2\Delta x^2 = A \\ B = -2/\Delta x^2 \end{cases}$$

- Thus: $\boxed{f''(x) \approx \dfrac{f\left(x+\Delta x\right) - 2f\left(x\right) + f\left(x-\Delta x\right)}{\Delta x^2}}$

- And: $f(1.5) \approx \dfrac{f\left(.6\right) - 2f\left(.5\right) + f\left(.4\right)}{(.1)^2} = \dfrac{1.672 - 2(1.5) + 1.368}{.01} = 4$

- Actual Value is 4!

2. Using Math Tables and Programs to Find FD Approximations

a. Math Tables
b. MATLAB Demo