Richardson Extrapolation for 2-Point FD Approximation

I. **Derivation:** Recall the 3-point central-difference approximation for the 1st derivative:

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} - \frac{h}{2}f''(\xi)$$

a. Let $D = f'(x)$ and $D_h = \frac{f(x_0+h)-f(x_0)}{h}$. The approximation above becomes:

$$D = D_h - \frac{h}{2}f''(\xi)$$

b. Let:

$$\frac{h}{2}f''(\xi) = \frac{h}{2}f''(x_0) + \frac{h}{2}[f''(\xi) - f''(x_0)]$$

c. Therefore:

$$D = D_h - \left[\frac{h}{2}f''(x_o) + \frac{h}{2}[f''(\xi) - f''(x_0)]\right] \rightarrow$$

$$D = D_h - \frac{h}{2}f''(x_o) - \frac{h}{2}f''(\xi) + \frac{h}{2}f''(x_0) \rightarrow$$

$$\frac{D - D_h + \frac{h}{2}f''(x_0)}{h} = -\frac{1}{2}f''(\xi) + \frac{1}{2}f''(x_0) \rightarrow$$

$$\left|\frac{D - D_h + \frac{h}{2}f''(x_0)}{h}\right| = \left|\frac{1}{2}f''(\xi) - \frac{1}{2}f''(x_0)\right|$$

d. Recall that $x_0 - h < \xi < x_0 + h$. Since $\lim_{h\to 0} \xi = x_0$,

$$\lim_{h\to 0}\left|\frac{D - D_h + \frac{h}{2}f''(x_0)}{h}\right| = \lim_{h\to 0}\left|\frac{1}{2}f''(\xi) - \frac{1}{2}f''(x_0)\right| = 0$$

e. Therefore:

$$D \rightarrow D_h - \frac{h}{2}f''(x_0)$$

f. Let $K_1 = -\frac{1}{2}f''(x_0)$. Rewriting the equation above:

$$D \to D_h + hK_1 \; or \; D = D_h + hK_1 + o(h) \to hK_1 = D - D_h + o(h)$$

Now if we have information halfway to x_0 i.e. at $x_0 + \frac{h}{2}$, we can say that:

$$D = D_{h/2} + \frac{h}{2}K_1 + o(h)$$

g. Now I consider the system of 2 equations and I have just created:

$$D = D_h + hK_1 + o(h)$$

$$D = D_{h/2} + \frac{h}{2}K_1 + o(h)$$

Subtracting yields:

$$0 = D_h - D_{h/2} + \frac{1}{2}hK_1 + o(h) \to$$

$$\frac{1}{2}hK_1 = D_{h/2} - D_h + o(h) \to$$

$$hK_1 = 2\left(D_{h/2} - D_h\right) + o(h)$$

h. Recall $hK_1 = D - D_h + o(h)$, therefore

$$D - D_h = 2\left(D_{h/2} - D_h\right) + o(h) \to$$

$$D = 2D_{h/2} - 2D_h + D_h + o(h) \to$$

$$D = 2D_{h/2} - D_h + o(h)$$

II.

Subsequent Refinements

1. General Formula:

$$\frac{b^p D_{h/b} - D_h}{b^p - 1}$$

Where: $b\ = step\ size\ factor$

$p = order\ of\ error\ (i.e. h^p)\ of\ p^{th}\ extrapolation$

   a. I.e for $b = 2$ (i.e. step size factors of $h/2$) the 2nd refinement of $o(h^2)$ is

$$\frac{4D_{h/2} - D_h}{3}$$

   b. I.e for $b = 2$ (i.e. step size factors of $h/2$) the 3nd refinement of $o(h^3)$ is

$$\frac{8D_{h/2} - D_h}{7}$$

   c. I.e for $b = 10$ (i.e. step size factors of $h/10$) the for $o(h^3)$ is

$$\frac{1000D_{h/100} - D_h}{999}$$