Lecture Notes

Section 6.2 - Numerical Differentiation

I) Deriving Estimate For 1st Derivative

- consider following data

x	$f(x)$
.9	2.4596
1	2.7182
1.1	3.0042

$f'(1) \approx \boxed{\frac{f(1)-f(.9)}{1-.9} = 2.5868}$

$f'(1) \approx \boxed{\frac{f(1.1)-f(1)}{(1.1)-1} = 2.860}$

what if I average these

$$\therefore \frac{1}{2}\left(\frac{f(1)-f(.9)}{.1} + \frac{f(1.1)-f(1)}{.1}\right)$$

$$= \boxed{\frac{f(1.1)-f(.9)}{.2} = 2.723}$$

WHICH IS THE BEST ANSWER?

II Derive $f'(x_0)$ Using Following Data

"2 PT. Forward Difference Estimate"

A) Consider Data $\Rightarrow f'(x_0)$

x	$f(x)$
x_0	$f(x_0)$
x_0+h	$f(x_0+h)$

Let
$$\begin{cases} f(x_0) = f(x_0) & \text{Duh!} \\ f(x_0+h) = \underbrace{f(x_0) + \frac{h}{1!}f'(x_0)}_{\text{Taylor Series Expansion}} + \frac{h^2}{2!}f''(\xi) \end{cases}$$

ERROR TERM

B) Ignore Error Term for a "second"

Can I combine $f(x)$ & $f(x_0+h)$ is such a way that $a f(x_0) + b f(x_0+h) = f'(x_0)$

$$a f(x_0) = a f(x_0)$$

$$b f(x_0+h) = \underbrace{b f(x_0)}_{=0} + \underbrace{b h f'(x_0)}_{= f'(x_0)} \qquad \text{(ignore error for now)}$$

i.e. $a f(x_0) + b f(x_0) = 0 \rightarrow a + b = 0$

$b h f'(x_0) = f'(x_0) \Rightarrow b = 1/h \Rightarrow a = -\frac{1}{h}$

$$\therefore\ -\frac{1}{h} f(x_0) + \frac{1}{h} f(x_0+h) = f'(x_0)$$

$$\Rightarrow \boxed{f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h}}$$

C) Now Determine Error

$$E = a(0) + b\left[\frac{h^2}{2!} f''(\xi)\right] = \frac{1}{h}\left(\frac{h^2}{2!} f''(\xi)\right)$$

$$\Rightarrow \boxed{E = \frac{h}{2!} f''(\xi)} \Rightarrow E \text{ has } O(h)$$

III ESTIMATE $f'(x_0)$ FOR FOLLOWING DATA

(3 pt centrale difference formula)

x	$f(x)$
x_0-h	$f(x_0-h)$
x_0	x_0
x_0+h	$f(x_0+h)$

$\rightarrow$ to find $f'(x_0)$

need 3rd term

$$\begin{cases} f(x_0-h) = f(x_0) - hf'(x_0) + \frac{h^2}{2!}f''(x_0) - \frac{h^3}{3!}f'''(\xi) \\ f(x_0) = f(x_0) \quad \text{"no error"} \\ f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}f'''(\xi) \end{cases}$$

$\Rightarrow$ Forget error term for a momen

$$\begin{cases} af(x_0-h) = af(x_0) - ahf'(x_0) + \frac{ah^2}{2!}f''(x_0) \\ bf(x_0) = bf(x_0) \\ cf(x+h) = cf(x_0) + chf'(x_0) + \frac{ch^2}{2!}f''(x_0) \end{cases}$$

sum down columns

$\downarrow \quad\quad \downarrow \quad\quad \downarrow$

$0 \quad\quad f'(x_0) \quad\quad 0$

$$\begin{cases} a+b+c=0 \\ ch-ah=1 \Rightarrow c-a=1/h \\ a+c=0 \Rightarrow \end{cases}$$

$$\frac{\begin{array}{r} c-a=1/h \\ a+c=0 \end{array}}{2c=1/h} \Rightarrow c=1/2h$$

$\Rightarrow a=-1/2h$

$\Rightarrow b=0$

$\Rightarrow f'(x_0) = -\frac{1}{2h} f(x_0-h) + \frac{1}{2h} f(x_0+h)$

$$\Rightarrow \boxed{f'(x_0) = \frac{f(x+h) - f(x_0-h)}{2h}}$$

B. Find Error:

$$E = a\left(-\frac{h^3}{3!} f'''(\xi)\right) + c\left(\frac{h^3}{3!} f'''(\xi)\right)$$

$$= -\frac{1}{2h}\left(-\frac{h^3}{3!} f'''(\xi)\right) + \frac{1}{2h}\left(\frac{h^3}{3!} f'''(\xi)\right)$$

$$= \frac{1}{2}\frac{h^2}{3!} f'''(\xi) + \frac{1}{2}\frac{h^2}{3!} f'''(\xi) = \boxed{\frac{h^2}{3!} f'''(\xi) = E}$$

$\Rightarrow E$ has $O(h^2)$

IV. ESTIMATE $f''(x_0)$ using data points:

x	$f(x)$
x_0-h	$f(x_0-h)$
x_0	$f(x_0)$
x_0+h	$f(x_0+h)$