## I. Demonstration - Intro to Optimal Points

a) consider the function

$$f(x) = (x-1)(x-.5)(x)(x+.5)(x+1) \quad \text{of} \quad x \in [-1,1]$$

or $f(x) \; x^5 - 1.25x^3 + .25x$

(i.e. a fifth order polynomial on $[-1,1]$ w/roots $(1, .5, 0, -.5, -1)$)

b) If pick any 6 random points along $f(x)$ I can come up w/ 5TH ORDER polynomial poly is an exact fit

$$\boxed{\text{ChIntro}(0, 5, 0, [-1,1])}$$

c) If I am limited to 5 points, I generate a 4TH ORDER interpolating poly that "does its best" to fit the function

$$\boxed{\text{ChIntro}(0, 4, 0, [-1,1])}$$

1000 RAN EXPERIMENT

↳ $\boxed{\text{ChIntro}(1, 4, 0, [-1,1])}$

d) Note the error measure

$$\boxed{|e|_\infty = |f(x) - P_n(x)|_\infty = \max(\text{abs}(f(x) - P_n(x)))}$$

e) It "turns out" the n optimal point on interval $[-1,1]$ are given by

$$X_j = \cos\left(\frac{2j-1}{2n}\pi\right) \qquad \boxed{\text{ChIntro}(2, 4, 0, [-1,1])}$$

i.e. for $n=5 \Rightarrow \cos\left(\frac{\pi}{10}\right), \cos\left(\frac{3\pi}{10}\right), \cos\left(\frac{5\pi}{10}\right), \cos\left(\frac{7\pi}{10}\right), \cos\left(\frac{9\pi}{10}\right.$

②

) What am I minimizing?

$$|f(x) - P_n(x)| = \left|\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)\right|$$

↑ error for interpolating polynomial

① f is given & n is fixed

② the only thing I can effect are the values $(x_0, \ldots x_n)$ in the expression

$$\omega(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$$

A) Some Definitions for function norms

$$\boxed{\|f\|_\infty = \max_{x\in[a,b]} |f(x)|}$$ where f continuous on $[a,b]$

$$\boxed{\|f\|_2 = \left(\int_a^b [f(x)]^2\,dx\right)^{1/2}}$$ where f continuous on $[a,b]$

B) wish to find $\{x_0, \ldots x_n\}$ that minimizes

$\|f(x) - P_n(x)\|_\infty$ (use Chebyshev Polys)

→ $\|f(x) - P_n(x)\|_2$

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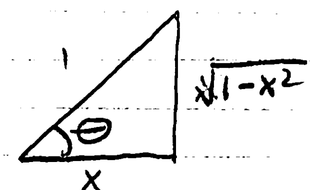
## III Chebyshev Polynomial

A. The optimal pts selected were roots of a $5^{th}$ order "Chebyshev Polynomial"

B) Definition: For Each non-neg integer $n$, the Chebyshev Poly $T_n$ on $[-1,1]$ is

$$\boxed{T_n(x) = \cos(n\underbrace{\cos^{-1}(x)}_{\theta})}$$




$$\boxed{\begin{aligned}\cos^{-1}(x) &= \theta \\ x &= \cos(\theta)\end{aligned}}$$

$$\Rightarrow T_n(x) = \cos(n\theta)$$

C) How is this a Polynomial?

① $T_{n+1}(x) = \cos((n+1)\theta) = \cos(n\theta + \theta)$
$= \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta)$

② $T_{n-1}(x) \qquad = \cos(n\theta)\cos(\theta) + \sin(n\theta)\sin(\theta)$

③ $T_{n+1}(x) + T_{n-1}(x) = 2\cos(n\theta)\cos(\theta)$
$= 2\,T_n(x)\,(x) = 2xT_n(x)$

$$\Rightarrow \boxed{T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)}$$

recurrence relation

④

d) initial expressions
$$\begin{cases} T_0(x) = \cos(0 * \cos^{-1}(x)) = \cos(0) = \boxed{1} \\ T_1(x) = \cos(1 * \cos^{-1}(x)) = \cos(\cos^{-1}(x)) = \boxed{X} \end{cases}$$

"recurrance relation"
$$\begin{cases} T_2(x) = 2xT_1 + T_0 = 2x^2 - 1 \\ T_3(x) = 2xT_2(x) - T_1(x) = 2x(2x^2-1) - x = \boxed{4x^3 - 3x} \\ T_3(x) = 8x^4 - 8x^2 + 1 \quad \text{(let class get)} \end{cases}$$

e) <u>In General</u>:

① $T_n(x)$ is an $n^{th}$-degree poly.

② For $n \geq 1$, leading coeff of $T_n$ is $2^{n-1}$

③ $T_n(x) = \begin{cases} \text{even} & \text{if } n = \text{even} \\ \text{odd} & \text{is } n = \text{odd} \end{cases}$

IV <u>Theorems</u>

A. Chebyshev Polynomial of Degree n has roots at

$$x_j = \cos\left(\frac{2j-1}{2n}\pi\right), \quad j = 1, \dots, n$$

Proof: $T_n(x_j) = \cos\left(n \cos^{-1}\left(\cos \frac{2j-1}{2n}\pi\right)\right)$
$$= \cos\left(\cancel{n}\left(\frac{2j-1}{2\cancel{n}}\pi\right)\right) = \cos\left(\frac{2j-1}{2}\pi\right) = 0$$

QED

B) Chebyshev Poly $T_n(x)$ will have min/max at

$z_j = \cos\left(\frac{j\pi}{n}\right)$

note

Proof: $T_n = \cos(n \cos^{-1}(\cos(\frac{j\pi}{n})))$

$= \cos(n(\frac{j\pi}{n})) = \cos(j\pi) \boxed{= (-1)^j}$

$\Rightarrow T_n' = -j\sin(j\pi) = 0$ since $j = 1, 2 \ldots n$

QED

i.e. min/max are $\pm 1$

V) As ASIDE - so FAR Everything has Been on Interval $[-1, +1]$, with optimal points at $x_j = \cos\left(\frac{2j-1}{2n}\pi\right)$

A) If $I$ is a general interval $[a, b]$

① center of interval $= \frac{a+b}{2}$

② "radius" of interval $= \frac{|a-b|}{2}$

$$\boxed{x_j = c + r\cos\left(\frac{2j-1}{2n}\pi\right)}$$

<u>Example</u>: Find Optimal $T_5$ points for $I = [-1, 3]$

$c = 1$
$r = 2$ $\Rightarrow$ $x_j = 1 + 2\cos\left(\frac{2j-1}{2n}\pi\right)$

$x_1 =$ ..... $x_2 =$ ...

Example for $f(x) = xe^{-x}$ ChIntro(3)62)

VI: Proving that Chebshev Polynomials Minimize $|f(x) - P_n(x)|_\infty$

A) Definition MONIC POLYNOMIALS - ALL POLYNOMIALS WHOSE LEADING COEFFICIENT IS ONE.

i.e. ① $[(x-x_0)](x-x_1) \dots (x-x_n)$ is monic

② $T_n(x)$ is not since leading term for $n>1$ is $2^{n-1}$, however I can create a monic set Chebshev polynomials

$$\tilde{T}_n(x) = \frac{1}{2^{n-1}} T(x) \quad \text{for } n>1$$

$\boxed{\text{note! } T(x) \;\&\; \tilde{T}(x) \text{ have the same roots}}$

B) Theorem: The monic polynomial $\tilde{T}_n(x)$ for $n>1$ Satisfies $\frac{1}{2^{n-1}} = \max_{x\in[-1,1]} |\tilde{T}_n(x)| \leq \max_{x\in[-1,1]} P_n(x)$

where $P_n \in \tilde{\Pi}_n$ (the set of all monic polynomials $(x-x_0)(x-x_1) \dots (x-x_n)$)

⑦

① Suppose $\max_{x\in[-1,1]} |p_n(x)| \leq \frac{1}{2^{n-1}} = \max_{x\in[-1,1]} |T(x)|$

② let $q(z_j) = \tilde{T}_n(z_j) - p_n(z_j)$ where $z_j$ represents to extrema of $\tilde{T}_n$, ie where $\tilde{T}_n = \pm\frac{1}{2^{n-1}}$

$\Rightarrow q(z_j) = 2^{1-n}(-1)^j - P_n(z_j)$

③ Since $\tilde{T}_n(z_j)$ & $P_n(z_j)$ are monic, ie (leading term is 1) of degree n, then $q(z_j)$ has degree of at most $n-1$

④ since $|p_n(x)| \leq \frac{1}{2^{n-1}}$ then

$$q(z_j) \leq \begin{cases} \leq 0 & n \text{ odd} \\ \geq 0 & n \text{ even} \end{cases}$$

⑤ therefore $q(z_j)$ has at least one root between $j$ & $j+1$ for $j = 0, 1, 2, 3, 4, \ldots, n$

⑥ I.E $q(z_j)$ has $n$ roots but has order of at most $n-1$. There fore by Fund Theorem of Alg $q(z_j) = 0$

⑦ $\therefore p_n = \tilde{T}_n$ (i.e $|P_n|$ can not be $< |\tilde{T}_n(x)|$)

⑧ Thus the roots of the Chebyshev polynomials minimizes $|f(x) - P_n(x)|_\infty$ QED