Lecture Notes

Section 5.4
Newtons Method

I) Example: (A MIRACLE?)

$(x_0, f_0)$ $(x_1, f_1)$ $(x_2, f_2)$

a) Consider the Data Points: $(-2,-3), (1,0), (2,5)$

b) Make the following table:

| $x_i$ | $f[x_i]$ | $f[x_i, x_{i+1}]$ | $f[x_i, x_{i+1}, x_{i+2}]$ |
|---|---|---|---|
| -2 | -3 | | |
| | | 1 | |
| 1 | 0 | | 1 |
| | | 5 | |
| 2 | 5 | | |

This is called a "divided difference table"

c) note: ① $f[x_i] = f_i$
"zeroeth divided difference"

② $f[x_i, x_{i+1}] = \dfrac{f_{i+1} - f_i}{x_{i+1} - x_i}$
"1st div. diff"

③ $f[x_i, x_{i+1}, x_{i+2}] = \dfrac{f_{i+2} - f_i}{x_{i+2} - x_i}$
"2nd div. diff"

d) Now Consider the circled values

$$f[x_0],\ f[x_0, x_1],\ f[x_0, x_1, x_2]$$

$$P_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$\Rightarrow P_2(x) = -3 + 1(x-(-2)) + 1(x-(-2))(x-1)$$

$$\Rightarrow P_2(x) = -3 + \cancel{x} + \cancel{2} + x^2 + 2x - \cancel{x} - \cancel{2} \Rightarrow \boxed{P_2(x) = x^2 + 2x - 3}\ !!$$

II <u>Add a point</u> to the dataset Find $P_3(x)$

$(-2,-3), (1,0), (2,5), (4,15)$

we can use the previous table:

"DIVIDED DIFFERENCE TABLE"

| | | | | |
|---|---|---|---|---|
| -2 | -3 | | | |
| | | 1 | | |
| 1 | 0 | | 1 | |
| | | 5 | | $\boxed{-1/6}$ |
| 2 | 5 | | $\boxed{0}$ | |
| | | $\boxed{5}$ | | |
| $\boxed{4}$ | $\boxed{15}$ | | | |

$$\Rightarrow P_3 = \underbrace{-3 + 1(x+2) + 1(x+2)(x-1)}_{\text{same as before}} - \tfrac{1}{6}(x+2)(x-1)(x-2)$$

$$\Rightarrow P_3 = -3 + \cancel{x} + \cancel{2} + x^2 + 2\cancel{x} - \cancel{x} - \cancel{2} - \tfrac{1}{6}x^3 + \tfrac{1}{6}x^2 + \tfrac{2}{3}x - \tfrac{2}{3}$$

$$\Rightarrow \boxed{P_3 = -\tfrac{1}{6}x^3 + \tfrac{7}{6}x^2 + \tfrac{8}{3}x - \tfrac{11}{3}}$$

$\Rightarrow$ <u>Advantage</u> of Newton over Legrange ...

one can add new points without starting from scratch

Example

III Consider Follow Data Points:

| $x$ | $f$ | | | |
|---|---|---|---|---|
| 0 | 3 | 2 | 0 | 0 |
| 1 | 5 | 2 | 0 | |
| 2 | 7 | 2 | | |
| 3 | 9 | | | |

what's going on

$$P_3 = 3 + 2(x-0) + 0(x-0)(x-1) + 0(x-0)(x-1)(x-2)$$

$$\Rightarrow \boxed{P_3(x) = 2x+3}$$

conclusion... Data was linear!!

Recall: Theorems guarantee a polynomial of "at most" $n$

IV Error

$$f(x) - P(x) = f[x_0, x_1, x_2 \ldots x_n, x] \prod_{x=0}^{n} (x - x_i)$$

looks like Legrange Error

... In fact

V Theorem: ① Let $x_0 \ldots x_n$ be $n+1$ distinct points on $[a,b]$

② If $f$ is continuous on $[a,b]$

③ w/ $n$ continuous derivatives on $(a,b)$

④ $\exists\ \xi$ on $(A, B)$

⑤ $\ni$ $f[x_0 \ldots x_0] = \frac{f^{(n)}(\xi)}{n!}$

A. Setting Up A Proof

① consider $f[x_0, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

looks like slope of secant line.

② By mean value theorem there $\exists$ a $\xi \in (x_0, x_1)$ such that $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(\xi)$

$\downarrow$

$\Rightarrow$ How might I extend this thought and prove

$f[x_0 \ldots x_n] = \dfrac{f^{(n)}(\xi)}{n!}$ ?