Lecture Notes

Section 5.1c
Interpolation Polynomials
(Theorems)

I **Theorem** - If $x_0, x_1, x_2 \ldots x_n$ are $n+1$ unique points

$\boxed{\text{Uniqueness}}$

- And f exists at each point
- Then $\exists$ a unique polynomial P of degree at most n such that

$$P(x_i) = f(x_i)$$

- P is called the interpolating polynomials

Proof:

A) Existence - we already proved existence

$$P_n(x) = \sum_{i=0}^{n} L_{n,i}(x) f_i$$

B) Uniqueness -

1. Assume there are 2 "different" interpolating polynomials P & Q for $x_0, x_1, \ldots, x_n$
2. Let $h(x) = P(x) - Q(x)$, therefore h is also a polynomial of degree n
3. $\therefore h(x_i) = P(x_i) - Q(x_i) = f_i - f_i = 0$
4. this means that an n^{th} degree polynomial $h(x_i)$ has $n+1$ roots
5. By "fundamental theorem of algebra," this is only possible of $h(x) = 0$
6. $\therefore P(x) - Q(x) = 0 \Rightarrow P(x) = Q(x)$ (contradiction)

QED

II **Theorem:** If ① $x_0, \dots x_n$ are distinct points on $[a,b]$

$\boxed{\text{ERROR}}$

② f is continuous on $[a,b]$

③ f has $n+1$ continuous derivatives on $[a,b]$

Then: for each $x \in [a,b]$ $\exists$ $\xi(x) \in [a,b]$ such that

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n)$$

where P is the interpolating polynomial

$\boxed{\text{Proof P 348 - 349}}$

III. Advantage/Disadvantages of Lagrange Form

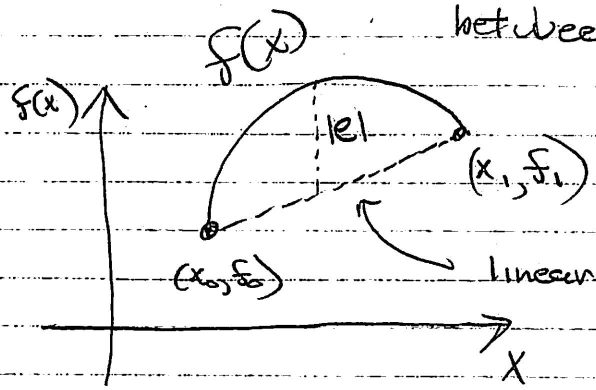
⊕ simplicity of derivation

⊕ theoretic value - techniques develope later bases on Lagrange form

⊖ as more data becomes available, must start work from scratch.

⊖ Lagrange polynomials very cumbersome to evaluate, integrate, differentiate, etc

IV: <u>Some</u> <u>Analysis</u>: ① Given the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$
② What is the maximum error if we perform a linear interpolation between the points



$\boxed{\text{note: } n=1 \text{ (one less than \# pts)}}$

$$① \quad |f(x) - P(x)| = \max_{x\in[x_0,x_1]} \left[\frac{f''(\xi)}{2!} (x-x_0)(x-x_1) \right] \quad (n+1=2)$$

$$= \frac{1}{2} \max_{x\in[x_0,x_1]} \underbrace{|(x-x_0)(x-x_1)|}_{} \max_{\xi\in[x_0,x_1]} |f''(\xi)|$$

find min/max of this expression

$$② \quad \frac{d}{dx}\left(x^2 - (x_0+x_1)x + x_0x_1 \right) = 0 \Rightarrow 2x - (x_0+x_1) = 0$$

$$\Rightarrow x = \left(\frac{x_0+x_1}{2} \right) \Rightarrow \text{critical point}$$

$$③ \quad (x-x_0)(x-x_1) = \left(\frac{x_0+x_1}{2} - x_0 \right)\left(\frac{x_0+x_1}{2} - x_1 \right)$$

$$= \left(\frac{x_1-x_0}{2} \right)\left(\frac{x_0-x_1}{2} \right) = -\frac{1}{4}(x_0-x_1)^2$$

$$④ \quad |f(x) - P(x)| = \frac{1}{2} \left| -\frac{1}{4}(x_0-x_1)^2 \right| \max_{\xi\in[x_0,x_1]} |f''(\xi)|$$

$$= \boxed{\frac{1}{8} [x_0-x_1]^2 \max_{\xi\in[x_0,x_1]} |f''(\xi)|} \quad \text{max error for linear int}$$

V: Max Error: Linear Interpolation when

$f(x) = e^x, \quad x_0 = 1, \quad x_1 = 2$ $\qquad f''(x) = e^x$

$$|f(x) - P(x)| \le \frac{1}{8}(x_0 - x_1)^2 \max_{\xi \in [1,2]} e^{\xi}$$

$$\le \frac{1}{8}(1)^2 e^2 = .92363$$

$\boxed{\text{Demo: Go 4}}$