Lecture Notes

Section 5.1
Legrange Polynomials

I. Linear Interpolation Revisited

ⓐ Given 2 fixed points (x_1, f_1) & (x_2, f_2)

ⓑ Devise a polynomial $P_1(x) = a_0 + a_1 x$

such that $\begin{cases} P_1(x_0) = a_0 + a_1 x_0 = f_0 \\ P_1(x_1) = a_0 + a_1 x_1 = f_1 \end{cases}$

$$\Rightarrow \begin{bmatrix} 1 & x_0 & f_0 \\ 1 & x_1 & f_1 \end{bmatrix} \xrightarrow{RREF} \quad a_0 = \frac{x_1 f_0 - x_0 f_1}{(x_1 - x_0)}, \quad a_1 = \frac{f_1 - f_0}{x_1 - x_0}$$

$$\therefore \quad P_1(x) = \frac{x_1 f_0 - x_0 f_1}{(x_1 - x_0)} + \frac{f_1 - f_0}{x_1 - x_0} x$$

$$\Rightarrow \boxed{P_1(x) = \frac{x_1 - x}{x_1 - x_0} f_0 + \frac{x - x_0}{x_1 - x_0} f_1}$$

note: $P(x_1) = f_1$ ✓✓
$P(x_0) = f_0$ ✓✓

Quick Rewrite for Structure $\Rightarrow$ $\boxed{P_1(x) = \frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_0}{x_1 - x_0} f_1}$

"$L_{1,0}(x)$" "$L_{1,1}(x)$"

"Legrande Polynomials"

note $L_{1,0}(x) = \begin{cases} 1 & x = x_0 \\ 0 & x = x_1 \end{cases}$ $L_{1,1} = \begin{cases} 0 & x = x_0 \\ 1 & x = x_1 \end{cases}$

II Definition of Legrange Polynomial $L_{n,j}(x)$

① Has degree 'n' associate with interpolation point x_j such that

$$\boxed{L_{n,j}(x_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}}$$

② The Polynomial that fits point $(x_0 \ldots x_n), (f_0 \ldots f_n)$ is

$$\boxed{P_n(x) = \sum_{i=0}^{n} L_{n,i}(x) f_i}$$

③ Furthermore

note "$x - x_j$" is missing

$$\boxed{L_{n,j} = \frac{(x-x_0)(x-x_1)\ldots(x-x_{j-1})(x-x_{j+1})\ldots(x-x_n)}{(x_j-x_0)(x_j-x_1)\ldots(x_j-x_{j-1})(x_j-x_{j+1})\ldots(x_j-x_n)}}$$

note: "$(x_j - x_j)$" is missing

④ or more compactly

$$\boxed{L_{n,j} = \prod_{i=0,\, i \neq j}^{n} \frac{x - x_i}{x_j - x_i}}$$

III) Example: Fit a Parabola to 3 Points

$$(-2, 3), \ (1, 0), \ (3, 4)$$
$$(x_0, f_0) \quad (x_1, f_1) \quad (x_2, f_2)$$

← 2nd degree poly is possible

$$\Rightarrow L_{2,0} = \frac{(x-1)(x-3)}{(-2-1)(-2-3)} = \frac{x^2-4x+3}{(-3)(-5)} = \frac{1}{15}(x^2-4x+3)$$

$$\Rightarrow L_{2,1} = \frac{\overset{(x+2)}{(x-(-2))}(x-3)}{(1-(-2))(1-3)} = \frac{x^2-x-6}{(3)(-2)} = -\frac{1}{6}(x^2-x-6)$$

$$\Rightarrow L_{2,2} = \frac{(x+2)(x-1)}{(3+2)(3-1)} = \frac{x^2+x-2}{(5)(2)} = \frac{1}{10}(x^2+x-2)$$

$$\therefore P_2(x) = \overset{\frac{1}{5}}{\frac{3}{15}}(x^2-4x+3) - \cancel{\frac{0}{6}(x^2-x-6)} + \overset{\frac{2}{5}}{\frac{4}{10}}(x^2+x-2)$$

$$= \frac{1}{5}x^2 - \frac{4}{5}x + \frac{3}{5} + \frac{2}{5}x^2 + \frac{2}{5}x - \frac{4}{5}$$

$$\Rightarrow \boxed{P_2(x) = \frac{3}{5}x^2 - \frac{2}{5}x - \frac{1}{5}}$$

CHECK AGAINST POINTS

$\Rightarrow$ Demo – $\boxed{\text{Go 2}}$

IV) Example: Fit 4TH ORDER POLY TO 5 RANDOM POINTS

$\Rightarrow$ Demo $\boxed{\text{Go3}}$

V. <u>An Aside</u> (Example)

Consider same 3 points from Ex III

$(-2,3), (1,0), (3,4)$

Find quadratic polynomial that fits all 3 pts

i.e. $ax^2 + bx + c = y$

$$\begin{array}{ll} (-2,3) & 4a - 2b + c = 3 \\ (1,0) & a + b + c = 0 \\ (3,4) & 9a + 3b + c = 4 \end{array} \Rightarrow \begin{bmatrix} 4 & -2 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & -2 & 1 & 3 \\ 1 & 1 & 1 & 0 \\ 9 & 3 & 1 & 4 \end{bmatrix} \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 0 & 0 & 3/5 \\ 0 & 1 & 0 & -2/5 \\ 0 & 0 & 1 & -1/5 \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} a = 3/5 \\ b = -2/5 \\ c = -1/5 \end{array} \Rightarrow \boxed{\frac{3}{5}x^2 - \frac{2}{4}x - \frac{1}{5} = y}$$

same polynomial as III