Lecture Notes

Section 3.8
Iterative Methods

## I. General Discussion

A. Why Iterative Methods?
- they typically require less operations than direct methods

B. Basic Concept: Wish to rewrite

$$A\vec{x} = \vec{b} \longrightarrow \vec{x} = T\vec{x} + c$$

iterative problem

C. How: Spliting methods:

① Find matrices M & N such that

(i) $\boxed{A = M - N}$

(ii) then $A\vec{x} = \vec{b} \Rightarrow (M-N)\vec{x} = \vec{b}$

$$\Rightarrow M\vec{x} = N\vec{x} + \vec{b}$$

$$\Rightarrow \vec{x} = M^{-1}N\vec{x} + M^{-1}\vec{b}$$

② Typically we call $T = M^{-1}N$ and $\vec{c} = M^{-1}\vec{b}$

$$\Rightarrow \boxed{\vec{x} = T\vec{x} + \vec{b}}$$

a) Sufficient condition for convergence

$$\boxed{\rho(T) < 1}$$

## II) Jacobi Iteration

Example:
$$\begin{cases} 5x_1 + x_2 + 2x_3 = 10 \\ -3x_1 + 9x_2 + 4x_3 = -14 \\ x_1 + 2x_2 - 7x_3 = -33 \end{cases}$$

Rewrite as
$$\begin{cases} x_1 = \frac{1}{5}(10 - x_2 - 2x_3) \\ x_2 = \frac{1}{9}(-14 + 3x_1 - 4x_3) \\ x_3 = -\frac{1}{7}(-33 - x_1 - 2x_2) \end{cases}$$

Let $x^0 = [0,0,0]^T$ initial guess

1st Iteration $\Rightarrow$ $x_1^{(1)} = 2$, $x_2^{(1)} = \frac{-14}{9}$, $x_3^{(1)} = \frac{33}{7}$

2nd Iteration $\Rightarrow$
$$\begin{cases} x_1^{(2)} = \frac{1}{5}\left(10 + \frac{14}{9} - 2\left(\frac{33}{7}\right)\right) = .425 \\ x_2^{(2)} = \frac{1}{9}\left(-14 + 3(2) - 4\left(\frac{33}{7}\right)\right) = -2.98 \\ x_3^{(3)} = -\frac{1}{7}\left(-33 - 2 - 2(-14/9)\right) = 4.56 \end{cases}$$

DEMO $\boxed{\text{GOJI}}$

$\rightarrow$ Tol = .0005

$\rightarrow$ Note: In Matrix For

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & -1/5 & -2/5 \\ 1/3 & 0 & -4/9 \\ 1/7 & 2/7 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ -14/9 \\ 33/7 \end{bmatrix}$$

$$\vec{x}^{(n+1)} = \vec{T} \quad \vec{x}^{(n)} + \vec{c}$$

$\Rightarrow$ If I can come up w/ $T$ & $\vec{c}$, I can easily program this.

CODE FOUND AT WEBSITE

Finding $T$ & $\vec{c}$ for Jacobi It.

① Let $D$ be the diagonal part of $A$

② $-L$ be the lower triangular "left-overs"

③ $-U$ be the upper triangular "left-overs"

(NOTE THIS $L$ & $U$ have nothing to do with $\underline{LU}$ factorization! In this case $A = D - L - U$)

$$A = \begin{bmatrix} 5 & 1 & 2 \\ -3 & 9 & 4 \\ 1 & 2 & -7 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -7 \end{bmatrix}; \quad L = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ -1 & -2 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Now Let $T_{jac} = D^{-1}[L+U]$ $\quad \vec{c}_{jac} = D^{-1}\vec{b}$

$$\Rightarrow T_{jac} = \begin{bmatrix} 1/5 & & \\ & 1/9 & \\ & & -1/7 \end{bmatrix} \begin{bmatrix} 0 & -1 & -2 \\ +3 & 0 & -4 \\ -1 & -2 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & -1/5 & -2/5 \\ +1/3 & 0 & -4/9 \\ +1/7 & +2/7 & 0 \end{bmatrix}}$$

$T$

$$c_{jac} = \begin{bmatrix} 1/5 & & \\ & 1/9 & \\ & & 1/7 \end{bmatrix} \begin{bmatrix} 10 \\ -14 \\ -33 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ -14/9 \\ -33/7 \end{bmatrix}}$$

$\vec{c}$

(same as previously shown!)

Note: If $\rho(T_{jac}) \geq 1$ there will be no convergence!!

## III Gauss-Seidel Iteration

$\Rightarrow$ Similar to Jacobi, except when I get a new value for 'x' I use it subsequent equations'

**Example** (using Jacobi Example)

$$\begin{aligned} 5x_1 + x_2 + 2x_3 &= 10 \\ -3x_1 + 9x_2 + 4x_3 &= -14 \\ x_1 + 2x_2 - 7x_3 &= -33 \end{aligned} \quad \Rightarrow \quad \begin{cases} x_1 = \frac{1}{5}(10 - x_2 - 2x_3) \\ x_2 = \frac{1}{9}(-14 + 3x_1 - 4x_3) \\ x_3 = -\frac{1}{7}(-33 - x_1 - 2x_2) \end{cases}$$

$\Rightarrow$ Let $x^0 = [0, 0, 0]^T$

$\Rightarrow$ 1st Iteration

use this for $x_1$

$$\begin{cases} x_1 = 2 \\ x_2 = \frac{1}{9}(-14 + 3(2) - 4(0)) = -8/9 \\ x_3 = -\frac{1}{7}(-33 - 2 - 2(-8/9)) = 299/63 \end{cases}$$

new $x_2$

ETC. DEMO $\boxed{\text{GoGS1}}$ ← note: take less iterations than Jacobi It!

Since we are using the new $x_1$ in the $x_2$ equation & the new $x_1, x_2$ in the $x_3$ equation

$$x_1 = \frac{1}{5}(10 - x_2 - 2x_3) = 2 - \frac{1}{5}x_2 - \frac{2}{5}x_3$$

$$x_2 = \frac{1}{9}\left(-14 + 6 - \frac{3}{5}x_2 - \frac{6}{5}x_3 - 4x_3\right) = \frac{-8}{9} - \frac{1}{15}x_2 - \frac{26}{45}x_3$$

$$x_3 = -\frac{1}{7}\left(-33 - 2 + \frac{1}{5}x_2 + \frac{2}{5}x_3 + \frac{16}{9} + \frac{2}{15}x_2 + \frac{52}{45}x_3\right)$$
$$= 299/63 - \frac{1}{21}x_2 - 2/9\, x_3$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & -1/5 & -2/5 \\ 0 & -1/15 & -26/45 \\ 0 & -1/21 & -2/9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ -8/9 \\ 299/63 \end{bmatrix}$$

$$\vec{x}^{(k+1)} = T\, \vec{x}^{(k)} + \vec{c}$$

## Getting T & $\vec{c}$ directly

① Find L, D, U as before:

$$A = \begin{bmatrix} 5 & 1 & 2 \\ -3 & 9 & 4 \\ 1 & 2 & -7 \end{bmatrix},\quad L = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ -1 & -2 & 0 \end{bmatrix},\quad D = \begin{bmatrix} 5 & & \\ & 9 & \\ & & -7 \end{bmatrix},\quad U = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

② $T = (D-L)^{-1}U = \begin{bmatrix} 5 & 0 & 0 \\ -3 & 9 & 0 \\ 1 & 2 & -7 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1/5 & 0 & 0 \\ 1/15 & 1/9 & 0 \\ 1/27 & 2/63 & -1/7 \end{bmatrix} \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & -1/5 & -2/5 \\ 0 & -1/15 & -26/45 \\ 0 & -1/21 & -2/9 \end{bmatrix}}$$

T

③ $\vec{c} = [D-L]^{-1}\vec{b} = \begin{bmatrix} 1/5 & 0 & 0 \\ 1/15 & 1/9 & 0 \\ 1/21 & 2/63 & -1/7 \end{bmatrix} \begin{bmatrix} 10 \\ -14 \\ -33 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ -8/9 \\ 299/63 \end{bmatrix}}$

$\vec{c}$

## IV. SOR (Successive Over Relaxation) Method

(A) Method
- Take the $x^{(k+1)}_{GS}$ from Gauss-Seidel
- Combine it as a weighted average with $x^{(k)}_{k}$, i.e.,

$$\boxed{x^{(k+1)}_{SOR} = (1-\omega)x^{(k)} + \omega x^{(k+1)}_{GS}}$$

(B) Does this work? DEMO!

DEMO

(a) USE SAME SYSTEM AS BEFORE (see page 1)

(b) GoJ1 & GoGS1 → REVIEWING J & GS Iteration

for tol = .0005

(c) Experiment w/ SOR

GoSOR1(.5), GoSOR1(1.0), .. (1.5), (1.4), (1.8),

etc ⇒ experiment for min iterations

(d) Go1 a Grand Comparison

## (E) Spliting The Matrix

Recall $x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}\right]$

$\therefore \underset{SOR}{x_i^{(k+1)}} = (1-\omega) x_i^{(k)} + \frac{\omega}{a_{ii}}\left[ \quad * \quad \right]$

(1) Let: $\boxed{M = \frac{1}{\omega} D - L}$, $\boxed{N = \left(\frac{1}{\omega} - 1\right) D + U}$

(2) $\boxed{T_{SOR} = (M)^{-1} N}$ $\boxed{\vec{c}_{SOR} = M^{-1} \vec{b}}$

## Some Theorems

I

A) If A is
(1) Real & Symmetric
(2) w/ positive diagonal elements
(3) positive definite

Then GS converges

Recall: A is Positive Definite if $z^T A z > 0$ for all non-zero vectors $\vec{z} \ni z \in \mathbb{R}^n$

+ $\boxed{\text{Others p 233-234}}$