Lecture Notes

Section 3.5
LU Factorization

I LU Factorization Philosophy

a) Can we "factor" a matrix such that solving the matrix $A\vec{x} = \vec{b}$ might
① easier
② more accurate
③ save steps

i.e analogy ⇒ can we factor
⇒ $x^3 - 5x^2 + 4x = 0$ in to a "simpler problem, i.e $x(x-4)(x-1) = 0$
⇒ no just solve $x=0$, $x-4=0$, $x-1=0$

b) we can do this with LU factorization
i.e. $A = LU$ where

"L" is lower triangular matrix - all elements above diagonal $= 0$

"U" is upper triangular matrix - all elements below diagonal $= 0$

II) Examples:

$$L = \begin{bmatrix} 1 & 0 \\ -1/2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 3 \\ 0 & 13/2 \end{bmatrix} \Rightarrow LU = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$$

but $L = \begin{bmatrix} 1/2 & 0 \\ -1/4 & 1/5 \end{bmatrix}$ $U = \begin{bmatrix} 4 & 6 \\ 0 & 65/2 \end{bmatrix}$ is also an LU factorization

d) ① LU factorizations are not unique!
② All non-singular matrices have LU factorizations

III) Example

Using LU Factorization to solve a system

Let ① $A\vec{x} = \vec{B} \Rightarrow LU\vec{x} = \vec{b} = L(U\vec{x}) = b$

② Let $U\vec{x} = \vec{z} \Rightarrow L\vec{z} = \vec{b}$

③ Solve for $\vec{z}$, then for $\vec{x}$

i.e. Let: $\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5 \\ -17 \end{bmatrix}$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ -1/2 & 1 \end{bmatrix}}_{L}\underbrace{\underbrace{\begin{bmatrix} 2 & 3 \\ 0 & 13/2 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}}}_{\vec{z}} = \underbrace{\begin{bmatrix} -5 \\ -17 \end{bmatrix}}_{\vec{b}}$$

$\Rightarrow L\vec{z} = \vec{b} \Rightarrow \begin{bmatrix} 1 & 0 \\ -1/2 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -5 \\ -17 \end{bmatrix}$

$\Rightarrow \boxed{z_1 = -5}, \; -\frac{1}{2}z_1 + z_2 = -17 \Rightarrow z_2 = -17 + \frac{1}{2}(-5) = \boxed{-\frac{39}{2} = z_2}$

$\Rightarrow U\vec{x} = \vec{z} \Rightarrow \begin{bmatrix} 2 & 3 \\ 0 & 13/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5 \\ -39/2 \end{bmatrix}$

$\Rightarrow 13/2 \; x_2 = -39/2 \Rightarrow \boxed{x_2 = -3}$

$2x_1 + 3x_2 = -5 \Rightarrow 2x_1 = -5 - 3x_2 \Rightarrow 2x_1 = 4 \Rightarrow \boxed{x_1 = 2}$

IV) How Do I Factor Matrix? - Example

A **Example:** A 2×2 Matrix

$$\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} \xrightarrow[R2 + \frac{1}{2} R_1]{\text{ROW REDUCE}} \begin{bmatrix} 2 & 3 \\ 0 & 13/2 \end{bmatrix}$$

REMEMBER THIS NUMBER = "multiplier"

⇒ for convenience we write negative of multiplier in place of zero

$$\Rightarrow \begin{bmatrix} 2 & 3 \\ (-\frac{1}{2}) & 13 \end{bmatrix}$$

$$\Rightarrow U = \begin{bmatrix} 2 & 3 \\ 0 & 13 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

multipliers below diagonal

ones on diagonal

B. **Example:** 3x3 Matrix (w/o pivoting)

$$\begin{bmatrix} 1 & 4 & 3 \\ 2 & 7 & 9 \\ 5 & 8 & -2 \end{bmatrix} \xrightarrow[R3-5R1]{R2-2R1} \begin{bmatrix} 1 & 4 & 3 \\ (2) & -1 & 3 \\ (5) & -12 & -17 \end{bmatrix} \xrightarrow[R3-12R2]{}$$

negative of multiplier

$$\begin{bmatrix} 1 & 4 & 3 \\ (2) & -1 & 3 \\ (5) & (12) & -53 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 12 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 3 \\ 0 & -1 & 3 \\ 0 & 0 & -53 \end{bmatrix}$$

$= A$ ✓✓

C. **GRAND Example:** Solve A 3x3 system, with permutations & pivoting

$$\begin{bmatrix} 1 & 4 & 3 \\ 2 & 7 & 9 \\ 5 & 8 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -10 \\ 9 \end{bmatrix}$$

1. Factoring Step Ex. 3.15 p195
2. Solving Step Ex. 3.16 p198

V) A Final note!

Factoring Step costs	$\frac{2}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n$	
+ Solving Step costs	$2n^2 - n$	
Total	$\frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n$	!!! Just as much as Gaussian Elimination

⇒ So why Bother?! (See paragraph 3, page 199)