Lecture Notes

Section 3.4b
Error Estimates

I Perturbations to A and $\vec{b}$

Let:
1. δA = Perturbations to A
2. $\delta \vec{b}$ = Perterbations to $\vec{b}$
3. $\vec{x} + \delta\vec{x}$ represent a solution to

$$(A+\delta A)(\vec{x}+\delta\vec{x}) = (\vec{b}+\delta\vec{b})$$

A Long Derivation

$$\Rightarrow A\vec{x} + A\delta\vec{x} + \vec{x}\delta A + \delta A\delta\vec{x} = \vec{b} + \delta\vec{b} \qquad \text{(FOIL)}$$

$\Rightarrow$ since $A\vec{x} = \vec{b} \Rightarrow$

$$A\delta\vec{x} + \delta A\vec{x} + \delta A\delta\vec{x} = \delta\vec{b}$$

$$\Rightarrow \delta\vec{x} = A^{-1}\left[\delta\vec{b} - \delta A\vec{x} - \delta A\delta\vec{x}\right]$$

$$\Rightarrow \|\delta\vec{x}\| = \|A^{-1}(\delta\vec{b} - \delta A\vec{x} - \delta A\delta\vec{x})\| \qquad \text{(Taking Norm)}$$

$$\Rightarrow \|\delta\vec{x}\| \le \|A^{-1}\|\,\|\delta\vec{b} - \delta A\vec{x} - \delta A\delta\vec{x}\| \qquad \text{(property 5 of matrix Norms)}$$

$$\Rightarrow \|\delta\vec{x}\| \le \|A^{-1}\|\left(\|\delta\vec{b}\| + \|\delta A\vec{x}\| + \|\delta A\delta\vec{x}\|\right) \qquad \text{(Application of triangle inequality)}$$

$$\Rightarrow \|\delta\vec{x}\| \le \|A^{-1}\|\left(\|\delta\vec{b}\| + \underbrace{\|\delta A\|\|\vec{x}\|} + \underbrace{\|\delta A\|\|\delta\vec{x}\|}\right)$$

Property 5 of matrix norms.

$$\Rightarrow \|\delta\vec{x}\| - \|A^{-1}\|\|\delta A\|\|\delta\vec{x}\| \le \|A^{-1}\|\left(\|\delta\vec{b}\| + \|\delta A\|\|\vec{x}\|\right)$$

"gathering $\|\delta\vec{x}\|$ terms

$$\Rightarrow \|\delta\vec{x}\| \le \left(\frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|\delta A\|}\right)\left(\|\delta\vec{b}\| + \|\delta A\|\,\|\vec{x}\|\right)$$

solve for $\|\delta\vec{x}\|$

STILL NOT DONE! :)

$$\|\delta\tilde{x}\| \leq \left(\frac{\|A\|\|A^{-1}\|}{1 - \|A\|^{-1}}\right)\left(\frac{\|\delta b\|}{\|A\|} + \frac{\|\delta A\|}{\|A\|}\|x\|\right)$$ algebraic manipulation

$$\Rightarrow \|\delta\tilde{x}\| \leq \left(\frac{\|A\|\|A^{-1}\|}{1 - \|A\|\|A^{-1}\|(\|\delta A\|/\|A\|)}\right)\left(\frac{\|\delta b\|}{\|A\|} + \frac{\|\delta A\|}{\|A\|}\|x\|\right)$$ more alge. manip

$$\Rightarrow \|\delta\tilde{x}\| \leq \frac{k(A)}{1 - k(A)(\|\delta A\|/\|A\|)}\left(\frac{\|\delta b\|}{\|A\|} + \frac{\|\delta A\|}{\|A\|}\|x\|\right)$$ def of $k(A) = \|A\|\|A^{-1}\|$

$$\Rightarrow \frac{\|\delta x\|}{\|x\|} \leq \frac{k(A)}{1 - k(A)(\|\delta A\|/\|A\|)}\left(\frac{\|\delta b\|}{\|A\|\|x\|} + \frac{\|\delta A\|}{\|A\|}\right)$$ divide by $\|x\|$

$$\Rightarrow \frac{\|\delta x\|}{\|x\|} \leq \frac{k(A)}{1 - k(A)(\|\delta A\|/\|A\|)}\left(\frac{\|\delta b\|}{\|Ax\|} + \frac{\|\delta A\|}{\|A\|}\right)$$ since $\|Ax\| \leq \|A\|\|x\|$ (property 5)

$$\therefore \boxed{\frac{\|\delta x\|}{\|x\|} \leq \frac{k(A)}{1 - k(A)(\|\delta A\|/\|A\|)}\left(\frac{\|\delta b\|}{\|b\|} + \frac{\|\delta A\|}{\|A\|}\right)}$$

How do we use this result?

note role condition number plays in bounding the error

II) <u>Example</u>: 3.13 on Page 185

III Rounding Errors introduced by Gaussian Elimination

$$\boxed{\frac{\|\delta x\|_\infty}{\|x\|_\infty} \le \frac{k_\infty(A)\cdot n\cdot 10^{1-t}}{1-k_\infty(A)\cdot n\cdot 10^{1-t}}}$$

n = size of system
t = # of digits in floating arith sys.

IV Example

Recall Example 3.5 (163)

$$A = \begin{bmatrix} 2 & 1 & 4 & -1 \\ 2 & -2 & -1 & 2 \\ 5 & 7 & 14 & -8 \\ 1 & 3 & 2 & 4 \end{bmatrix}$$ in $F(10, 4, m, M)$ ('t')

$t=4$, $n=4$.

$\|A\|_\infty = 34$

"Matlab" ⇒ $\|\tilde{A}\|_\infty = 1.814$

$k_\infty = \|A\|_\infty \|A^{-1}\|_\infty = 61.67$

GEDEMS

$$\frac{\|\delta x\|_\infty}{\|x\|_\infty} \le \frac{61.67\times 4\times 10^{-3}}{1-61.67\times 4\times 10^{-3}} = \frac{.2467}{.7533} = .3275$$

matlab Example ⇒ in example

$\tilde{x} = [1.131, -.7928, .8500, -.9987]^T$
$x = [1, -1, 1, -1]$
$\delta x = [.131, .2072, -.1500, .0013]$

$$\Rightarrow \frac{\|\delta x\|_\infty}{\|x\|_\infty} = \frac{.2072}{1} \le .3275$$