Lecture Notes

Section 3.4a
Error Estimates / Condition #
(Day I)

I Error vs Residual: Consider the Linear System

$$A\vec{x} = \vec{b}$$

A. Let: ① $\vec{x}$ be an exact solution
② $\tilde{x}$ be a solution "infused" with error

B. Definition: ① $\vec{e} = \tilde{x} - \vec{x}$ "error"
② $\vec{r} = A\tilde{x} - \vec{b}$ "residual"

C. Example: Consider the System

$$\begin{bmatrix} 1 & -2 \\ -.99 & 1.99 \end{bmatrix} \vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \text{let } \tilde{x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\vec{e} = \tilde{x} - \vec{x} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$ "big error" $\Rightarrow \|e\|_\infty = 2$

$$\vec{r} = A\tilde{x} - \vec{b} = \begin{bmatrix} -1 \\ .99 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -.01 \end{bmatrix} \Rightarrow \|\vec{r}\|_\infty = .01$$

small residual

$$\Rightarrow \frac{\|\vec{e}\|_\infty}{\|\vec{r}\|_\infty} = 200!!$$ "error is 200 times bigger then residual"

II. <u>Theorem</u>: <u>Let</u>: ① A be non singular matrix

② $\tilde{x}$ be approximate sol'n to $A\vec{x} = \vec{b}$

③ $\vec{r} = A\tilde{x} - b$

④ $\vec{e} = \tilde{x} - \vec{x}$ ⑤ $\|\cdot\|$ be any natural matrix norm

<u>Then</u>:

① $\frac{1}{\|A\|}\|\vec{r}\| \le \|\vec{e}\| \le \|A^{-1}\|\|\vec{r}\|$

② $\frac{1}{\|A\|\|A^{-1}\|}\frac{\|\vec{r}\|}{\|\vec{b}\|} \le \frac{\|\vec{e}\|}{\|\vec{x}\|} \le \|A\|\|A^{-1}\|\frac{\|\vec{r}\|}{\|\vec{b}\|}$

note: $\|b\| \ne 0$, $\|x\| \ne 0$

A. <u>Proof</u> (of part 1) → ① $\vec{r} = A\tilde{x} - \vec{b} = A\tilde{x} - A\vec{x} = A(\tilde{x} - \vec{x}) = A\vec{e}$

② ∴ $\vec{e} = A^{-1}\vec{r}$

③ ∴ $\|\vec{e}\| = \|A^{-1}\vec{r}\| \le \|A^{-1}\|\|\vec{r}\|$ (prop 5 mat norm)

④ $\|\vec{r}\| = \|A\vec{e}\| \le \|A\|\|\vec{e}\| \Rightarrow \frac{1}{\|A\|}\|\vec{r}\| \le \|\vec{e}\|$

⑤ $\frac{1}{\|A\|}\|\vec{r}\| \le \|\vec{e}\| \le \|A^{-1}\|\|\vec{r}\|$ QED

III. <u>Definition</u>: Condition Number → $\boxed{K(A) = \|A\|\|A^{-1}\|}$

Example: $A = \begin{bmatrix} 1 & -2 \\ -.99 & 1.99 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} 199 & 200 \\ 99 & 100 \end{bmatrix}$

$\|A\|_\infty = \max(1+2, .99+1.99) = 3$

$\|A^{-1}\|_\infty = \max(199+200, 99+100) = 399$

$K(A) = (3)(399) = \boxed{1197}$

## IV. Rewrite Part 2 of Theorem

$$\boxed{\frac{1}{K(A)} \frac{\|\vec{r}\|}{\|\vec{b}\|} \le \frac{\|\vec{e}\|}{\|\vec{x}\|} \le K(A) \frac{\|\vec{r}\|}{\|\vec{b}\|}}$$

Therefore ① small $K(A)$ provides a good measure of error in an approximate solution, ② large $K(A)$ provides bad....

A. I.E. in above example

$$\frac{1}{1197} \frac{\|\vec{r}\|}{\|\vec{b}\|} \le \frac{\|\vec{e}\|}{\|\vec{x}\|} \le 1197 \frac{\|\vec{r}\|}{\|\vec{b}\|}$$

i.e. very bad indicator

B. Lower Bound For $K(A)$

$$1 = \|I\| = \|AA^{-1}\| \le \|A\|\|A^{-1}\| = K(A)$$