Lecture Notes

Section 3.3b
Matrix Norms

I **Definition** A "matrix norm" is a function $\|\cdot\|: \mathbb{R}^{n\times n} \Rightarrow \mathbb{R}$ (i.e. turns a matrix into a scalar), with the following properties

"so far like the vector norm"

1) $\|A\| \geq 0$
2) $\|A\| = 0$ iff $A = 0$ (the zero matrix)
3) $\|\alpha A\| = |\alpha| \|A\|$
4) $\|A+B\| \leq \|A\| + \|B\|$

& $\Rightarrow$ 5) $\|AB\| \leq \|A\| \|B\|$

II **Example**: Is a determinant a matrix norm?

① $\det|A|: \mathbb{R}^{n\times n} \Rightarrow \mathbb{R}$, but

② consider $A = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix}$

$\det(A) = (1)(6) - (2)(3) = 0$, but $A \neq 0$

$\therefore$ violates property 2!!

III **Definition**: Let $\|\cdot\|_v$ be a vector norm, then

$$\boxed{\|A\| = \max_{\|\vec{x}\|_v \neq 0} \frac{\|Ax\|_v}{\|x\|_v}}$$

called the "natural" or "operator" norm.

$\Rightarrow$

It follows that

$$\|A\| \geq \frac{\|A\vec{x}\|_V}{\|\vec{x}\|_V} \longrightarrow \boxed{\|A\vec{x}\|_V \leq \|A\|\|\vec{x}\|_V}$$

this is called the "consistency property"

IV. Theorem: Let $\|\cdot\|_V$ be a vector norm. The natural norm associated w/ $\|\cdot\|_V$ is a matrix norm

Property 1: $\|A\| = \max\limits_{\|x\|_V \neq 0} \frac{\|A\vec{x}\|_V}{\|\vec{x}\|_V} \geq 0$

$\Rightarrow$ since $\|\cdot\|_V$ is a vector norm both $\|A\vec{x}\|_V$ & $\|\vec{x}\| \geq 0$,

Property 2 $\|A\| = 0$ iff $A = 0$

$$\Rightarrow \|A\| = \max_{\|x_V\| \neq 0} \frac{\|A\vec{x}\|}{\|\vec{x}\|} = 0 \Rightarrow \|A\vec{x}\|_V = 0 \quad \text{for } \vec{x} = \vec{0}$$

$$\Rightarrow A\vec{x} = 0 \Rightarrow A = 0$$

Property 3: $\|\alpha A\| = \max\limits_{\|\vec{x}\|_V \neq 0} \frac{\|\alpha A\vec{x}\|_V}{\|\vec{x}\|_V} = \max\limits_{\|\vec{x}\|_V \neq 0} \frac{|\alpha|\|A\vec{x}\|_V}{\|\vec{x}\|_V}$

$$= |\alpha| \max_{\|x\|_V \neq 0} \frac{\|A\vec{x}\|_V}{\|\vec{x}\|_x} = |\alpha|\|A\| \checkmark$$

Property 4: $\|(A+B)\vec{x}\|_V = \|A\vec{x} + B\vec{x}\|_V \leq \|A\vec{x}\|_V + \|B\vec{x}\|_V$

triangle inequality of vector norms

$\Rightarrow \|A\vec{x}\| \leq \|A\|\|\vec{x}\|_V, \; \|B\vec{x}\| \leq \|B\|\|\vec{x}\|_V$

consistency property

$$\therefore \|(A+B)\vec{x}\|_V \leq (\|A\| + \|B\|)\|\vec{x}\|_V$$

$$\Rightarrow \frac{\|(A+B)\vec{x}\|_V}{\|\vec{x}\|_V} \leq \|A\| + \|B\| \Rightarrow \|A+B\| \leq \|A\| + \|B\|$$

Property 5 ⇒ Left for H.W.

V Example of Matrix Norm: $\boxed{\|A\|_\infty}$

① take absolute value of all elements in A

② sum up rows.

③ largest value = $\|A\|_\infty$

NOTE: $\|A\|_1$ is like $\|A\|_\infty$ except perform operation on column

i.e. $A = \begin{bmatrix} -1 & -2 \\ 3 & -4 \end{bmatrix}$

$$\Rightarrow \|A\|_\infty = \max\left(1+|-2|, 3+|-4|\right) = \max(3,7) = 7$$

VI Example: $\boxed{\|A\|_2}$

① Find A^TA

② Find eigenvalues of A^TA

③ take square root of largest eig val

$$A^TA = \begin{bmatrix} 1 & 3 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 10 & -14 \\ -14 & 20 \end{bmatrix}$$

$$\Rightarrow \det\begin{bmatrix} 10-\lambda & -14 \\ -14 & 20-\lambda \end{bmatrix} = 0 \Rightarrow 200 - 30\lambda + \lambda^2 - 196 = 0$$

$$\Rightarrow \lambda^2 - 30\lambda + 4 = 0$$

$$\Rightarrow \lambda = \frac{30 \pm \sqrt{900-4}}{2} \Rightarrow \lambda_{max} = 29.967$$

$$\Rightarrow \|A\|_2 = (29.967)^{1/2} = 5.474$$