

I Definition A "matrix norm" is a function  $\|\cdot\|: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$  (i.e. turns a matrix into a scalar), with the following properties

- "So far like the vector norm"
- (1)  $\|A\| \geq 0$
  - (2)  $\|A\| = 0$  iff  $A = 0$  (the zero matrix)
  - (3)  $\|\alpha A\| = |\alpha| \|A\|$
  - (4)  $\|A+B\| \leq \|A\| + \|B\|$

§  $\Rightarrow$  (5)  $\|AB\| \leq \|A\| \|B\|$

II Example: Is a determinant a matrix norm?

(1)  $\det|A|: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ , but

(2) consider  $A = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix}$

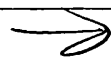
$\det(A) = (1)(6) - (2)(3) = 0$ , but  $A \neq 0$

$\therefore$  violates property 2!!

III Definition: Let  $\|\cdot\|_v$  be a vector norm, then

$$\|A\| = \max_{\|x\|_v \neq 0} \frac{\|Ax\|_v}{\|x\|_v}$$

called the "natural" or "operator" norm.



It follows that

$$\|A\| \geq \frac{\|A\vec{x}\|_V}{\|\vec{x}\|_V} \rightarrow \boxed{\|A\vec{x}\|_V \leq \|A\| \|\vec{x}\|_V}$$

this is called the "consistency property"

IV. Theorem: Let  $\|\cdot\|_V$  be a vector norm. The natural norm associated w/  $\|\cdot\|_V$  is a matrix norm

Property 1:  $\|A\| = \max_{\|\vec{x}\|_V \neq 0} \frac{\|A\vec{x}\|_V}{\|\vec{x}\|_V} \geq 0$

$\Rightarrow$  since  $\|\cdot\|_V$  is a vector norm both  $\|A\vec{x}\|_V \geq 0$  &  $\|\vec{x}\|_V \geq 0$

Property 2  $\|A\| = 0$  iff  $A = 0$

$$\Rightarrow \|A\| = \max_{\|\vec{x}\|_V \neq 0} \frac{\|A\vec{x}\|_V}{\|\vec{x}\|_V} = 0 \Rightarrow \|A\vec{x}\|_V = 0 \text{ for } \vec{x} = \vec{0}$$
$$\Rightarrow A\vec{x} = \vec{0} \Rightarrow A = 0$$

Property 3:  $\|\alpha A\| = \max_{\|\vec{x}\|_V \neq 0} \frac{\|\alpha A\vec{x}\|_V}{\|\vec{x}\|_V} = \max_{\|\vec{x}\|_V \neq 0} \frac{|\alpha| \|A\vec{x}\|_V}{\|\vec{x}\|_V}$ 
$$= |\alpha| \max_{\|\vec{x}\|_V \neq 0} \frac{\|A\vec{x}\|_V}{\|\vec{x}\|_V} = |\alpha| \|A\| \checkmark$$

Property 4:  $\|(A+B)\vec{x}\|_V = \|A\vec{x} + B\vec{x}\|_V \leq \|A\vec{x}\|_V + \|B\vec{x}\|_V$

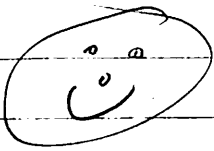
$\uparrow$   
triangle inequality  
of vector norms

$$\Rightarrow \|A\vec{x}\|_V \leq \|A\| \|\vec{x}\|_V, \|B\vec{x}\|_V \leq \|B\| \|\vec{x}\|_V$$

$\cdot$  consistency property

$$\therefore \|(A+B)\vec{x}\|_V \leq (\|A\| + \|B\|) \|\vec{x}\|_V$$

$$\Rightarrow \frac{\|(A+B)\vec{x}\|_V}{\|\vec{x}\|_V} \leq \|A\| + \|B\| \Rightarrow \|A+B\| \leq \|A\| + \|B\|$$

Property 5  $\Rightarrow$  Left for HW. 

## III Example of Matrix Norm: $\|A\|_{\infty}$

① take absolute value of all elements in A

② sum up rows,

③ largest value =  $\|A\|_{\infty}$

c.e.  $A = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$

NOTE:  $\|A\|_1$  is like  $\|A\|_{\infty}$  except perform operation on column

$$\Rightarrow \|A\|_{\infty} = \max(1 + |2|, 3 + |-4|) = \max(3, 7) = \underline{\underline{7}}$$

## IV Example: $\|A\|_2$

① Find  $A^T A$

② Find eigenvalues of  $A^T A$

③ take square root of largest eig val

$$A^T A = \begin{bmatrix} 1 & 3 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 10 & -14 \\ -14 & 20 \end{bmatrix}$$

$$\Rightarrow \det \begin{bmatrix} 10-\lambda & -14 \\ -14 & 20-\lambda \end{bmatrix} = 0 \Rightarrow 200 - 30\lambda + \lambda^2 - 196 = 0$$
$$\Rightarrow \lambda^2 - 30\lambda + 4 = 0$$

$$\Rightarrow \lambda = \frac{30 \pm \sqrt{900 - 4}}{2} \Rightarrow \lambda_{\max} = 29.967$$

$$\Rightarrow \|A\|_2 = (29.967)^{1/2} = \underline{\underline{5.474}}$$

