Lecture Notes

Section 3.3
Vector & Matrix Norms

I **Vector Norm**

In the simplest of terms a vector norm is any operation that reduces a vector to a scaler.

i.e. $\|\vec{v}\| : \mathbb{R}^n \longrightarrow \mathbb{R}$

However it must have the following properties

II **Properties** of $\|\bullet\|$ ~~for~~ for a values $\vec{x}, \vec{y} \in \mathbb{R}^n$

"general vector"

and α:

1. $\|\vec{x}\| \geq 0$
2. $\|\vec{x}\| = 0$ iff $\vec{x} = \vec{0}$ "zero vector"
3. $\|\alpha\vec{x}\| = |\alpha| \, \|\vec{x}\|$
4. $\|\vec{x}+\vec{y}\| \leq \|x\| + \|y\|$ ← triangle inequality

a) ℓ_1-norm $\quad \|\vec{x}\|_1 = \sum_{i=1}^{n} |x_i|$

b) ℓ_2-norm $\quad \|\vec{x}\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$ i.e "length"

c) ℓ_a-norm $\quad \|\vec{x}\|_a = \left(\sum_{i=1}^{n} |x_i|^a\right)^{1/a}$

d) ℓ_∞-norm $\quad \|\vec{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$

why does this follow from a, b, c

III Proof that $\|\vec{x}\|_2$ satisfies properties of norms

① $\|\vec{x}\|_2 = (x_1^2 + x_2^2 \dots x_n^2)^{1/2} > 0$ unless

$x_1 = x_2 = \dots x_n = 0 \Rightarrow$ in which case $\|\vec{x}\| = 0$

$\therefore \|\vec{x}\|_2 \geq 0$

② $\|\vec{x}\|_2 = 0$ iff $x_1 = x_2 \dots x_n = 0$,
$\therefore \|\vec{x}_2\| = 0$ iff $\vec{x} = \vec{0}$

③ $\|\alpha\vec{x}\|_2 = (\alpha^2 x_1^2 + \alpha^2 x_2^2 \dots \alpha^2 x_n^2)^{1/2}$
$= |\alpha| (x_1^2 + x_2^2 \dots x_n^2)^{1/2} = \alpha \|\vec{x}\|$

④ $\|\vec{x} + \vec{y}\|_2^2 = \sum (x_i + y_i)^2$
$= \sum x_i^2 + 2\sum x_i y_i + \sum y_i^2$

$\leq \|x\|_2^2 + 2\left|\sum x_i y_i\right| + \|y\|_2^2$

note I presume here
that $\sum x_i y_i \leq |\sum x_i y_i|$

why?

$\leq \|x\|_2^2 + 2\|x\|_2\|y\|_2 + \|y\|_2^2$

I presume here
$|\sum x_i y_i| \leq \|x\|_2 \|y\|_2$
Cauchy-Buniakowski Schwarz
Inequality (Proof on 172-173)

$= (\|x\|_2 + \|y\|_2)^2$

$\Rightarrow \therefore \|\vec{x} + \vec{y}\|_2 \leq \|x\|_2 + \|y\|_2$

IV Proof of Cauchy-Buniakowski-Schwarz Inequality

Let $\vec{x},\vec{y}\in\mathbb{R}^n$, Then $\left|\sum_{i=1}^{n} x_i y_i\right| \le \|\vec{x}\|_2 \|\vec{y}\|_2$

① If $\vec{x}=\vec{0}$ or $\vec{y}=\vec{0}$ proof is trivial i.e. $0=0$

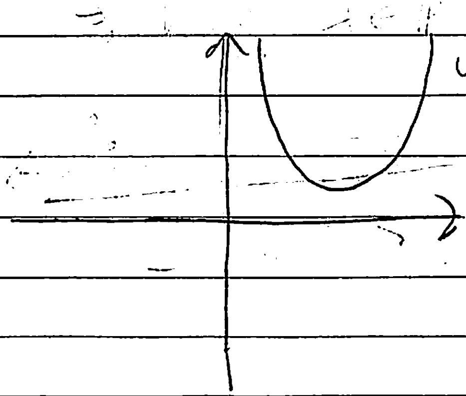
② Let $\lambda\in\mathbb{R}$,

$$0\le \|\vec{x}+\lambda\vec{y}\|_2^2 = \sum (x_i+\lambda y_i)^2$$
$$= \sum x_i^2 + 2\lambda\sum x_i y_i + \lambda^2 \sum y_i^2$$
$$= \|\vec{x}\|_2^2 + 2\lambda \sum x_i y_i + \lambda^2\|\vec{y}\|_2^2$$

③ Let $c=\|\vec{x}\|_2^2$, $b=\sum x_i y_i$, $a=\|\vec{y}\|_2^2$

$\Rightarrow$ $c + 2\lambda b + \lambda^2 a \ge 0$ or $a\lambda^2 + 2\lambda b + c \ge 0$

$\Rightarrow$ graphically this is parabola with one or no roots



i.e. $a\lambda^2+2\lambda b + c>0$

$\therefore$ λ is complex

$\therefore$ $\lambda = \frac{-2b\pm\sqrt{4b^2-4ac}}{2} \Rightarrow$

$4b^2-4ac\le 0 \Rightarrow b^2\le ac$

$\therefore$ $\left(\sum x_i y_i\right)^2 \le \|\vec{x}\|_2^2\|y\|_2^2$

$\Rightarrow$ $\boxed{\sum x_i y_i \le \|x\|_2 \|y\|_2}$