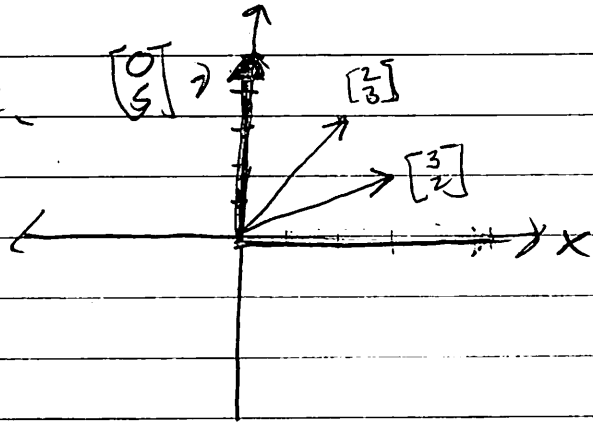
Lecture Notes

Chapter 3.1
Gaussian Elimination

I Expressing Linear Systems in Matrix Form

(a) Consider $\begin{cases} 2x+3y=0 \\ 3x+2y=5 \end{cases} \Rightarrow \begin{bmatrix} 2 \\ 3 \end{bmatrix} x + \begin{bmatrix} 3 \\ 2 \end{bmatrix} y = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$

i.e.



"i.e can I combine $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ + $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ in some combination to get $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$"

(b)

in Matrix Form

$$\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

(c) In augmented matrix form

$$\left[\begin{array}{cc:c} 2 & 3 & 0 \\ 3 & 2 & 5 \end{array}\right]$$

II Gaussian Elimination:

use "standard row ops" to reduce A to a upper triangular matrix

Standard Row Ops

1. Excange Rows
2. Multiply Row By Non-zero constant
3. Add a multiple of one row to another

i.e. Pivot

$$\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 3 & 2 & 5 \end{array}\right] \xrightarrow{R1/2} \left[\begin{array}{cc|c} 1 & 1.5 & 0 \\ 3 & 2 & 5 \end{array}\right] \xrightarrow[R2-3R1]{}$$

$$\rightarrow \begin{bmatrix} 1 & 1.5 & 0 \\ 0 & -2.5 & 5 \end{bmatrix}$$

Pivot

Now Solve by Back Substitution

$$-2.5y = 5 \Rightarrow \boxed{y = -2}$$
$$x + 1.5y = 0 \Rightarrow x - 3 = 0 \Rightarrow \boxed{x = 3}$$

Minor Variation - Don't Bother Turning Pivots into 1's

$$\begin{bmatrix} 2 & 3 & 0 \\ 3 & 2 & 5 \end{bmatrix} \xrightarrow{R2-\frac{3}{2}R1} \begin{bmatrix} 2 & 3 & 0 \\ 0 & -2.5 & 5 \end{bmatrix}$$

$$-2.5y = 5 \Rightarrow \boxed{y = -2}$$
$$2x + 3y = 0 \Rightarrow 2x - 6 = 0 \Rightarrow \boxed{x=3}$$

Counting Steps $\Rightarrow$ For a $n \times n$ matrix this takes $\frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n$ arithmetic operations

i.e. ($n=2 \Rightarrow 9$ steps, $n=3$ 28 steps, $n=10$ 805 steps)

III Gauss - Jordan Elimination

Converts Matrix A to a diagonal matrix using Row Ops.

Pivot $\rightarrow \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 3 & 2 & 5 \end{array}\right] \xrightarrow{R2 - \frac{3}{2}R1} \left[\begin{array}{ccc} 2 & 3 & 0 \\ 0 & -\frac{5}{2} & 5 \end{array}\right]$

Pivot

$\xrightarrow{R1 + \frac{6}{5}R2} \left[\begin{array}{cc|c} 2 & 0 & 6 \\ 0 & -5/2 & 5 \end{array}\right]$

$2x_1 = 6$
$-\frac{5}{2}y = 5 \Rightarrow \boxed{\begin{array}{l} x = 3 \\ y = -2 \end{array}}$

However this takes $n^3 + n^2 - n$ arith. ops for an $n \times n$ system

(i.e. $n=2$ 10 steps, $n=3$ 33 steps, $n=10$ 1090 steps)

IV Using Inverses:

$$Ax = b \Rightarrow x = A^{-1}b$$

$\Rightarrow$ to find inverse & calculate $A^{-1}b$ takes $2n^3$ operations

(i.e. $n=2 \Rightarrow 16$ ops, $n=3 \Rightarrow 54$ ops, $n=10 \Rightarrow 2000$ ops)

⇒ It is a goal of an applied mathematician to reduce arithmetic operations. I have worked w/ 10000 * 10000 systems.

Consider the ops req'd to solve

Gaussian Elimination = 6.69×10^{11} ops
Gauss-Jordan Elimination = 1.00×10^{12} ops
Inverse Matrices = 2.00×10^{12} ops

WHICH would you choose