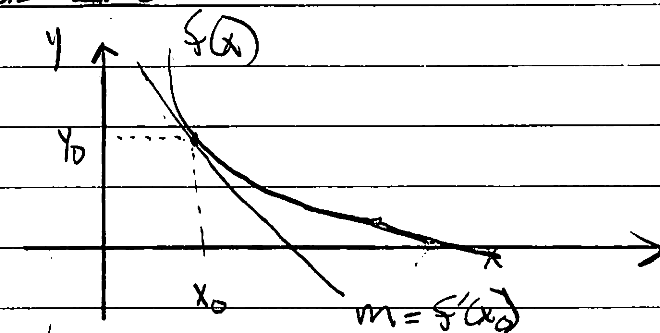
I Recall Pt-Slope Formula For Line

$$y - y_0 = m(x - x_0)$$



If (x_0, y_0) on curve $f(x)$ than equation of tangent line is:

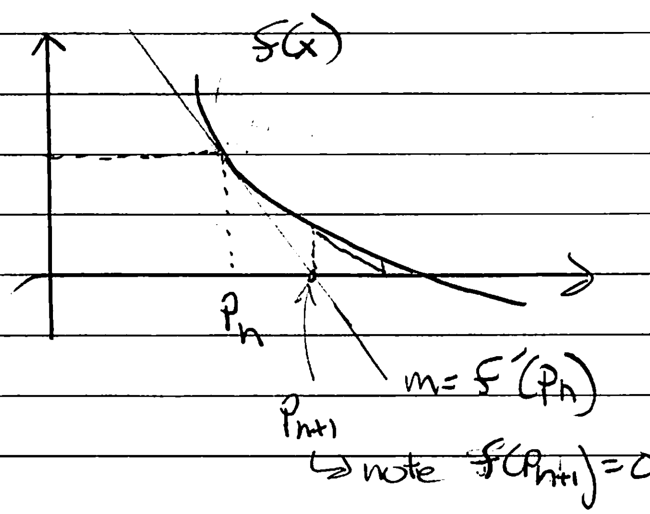
$$y - y_0 = f'(x_0)(x - x_0)$$

II Turn this Into in Iteration Formula

$$0 - f(p_n) = f'(p_n)(p_{n+1} - p_n)$$

$$\Rightarrow \frac{-f(p_n)}{f'(p_n)} = p_{n+1} - p_n$$

$$\Rightarrow \boxed{p_n = p_{n+1} - \frac{f(p_n)}{f'(p_n)}}$$



Newtons Method

III Newton's Method

Fixed point iteration scheme based on iteration function

$$\boxed{g(x) = x - \frac{f(x)}{f'(x)}}$$

****
* concerning *
* f(x)=0!! *
* then what? *

IV) Convergence Analysis

Note: Newtons Method is FP algorithm

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Therefore:

If: ① $g(x)$ continuous on $[a,b]$
② $g'(x)$ differentiable on (a,b)
③ $g: [a,b] \to [a,b]$
④ $|g'(x)| \leq k < 1$ for a $x \in [a,b]$

We can apply conclusions of FP Theorems

V Theorem

Let: ① f be twice differentiable on $[a,b]$
② $p \in [a,b]$
③ $f(p) = 0$ "condition at fixed point for newton's method"
④ $f'(p) \neq 0$

Then: $\exists\ \delta > 0 \ni$ for any $p_0 \in I = [p-\delta, p+\delta]$, The sequence p_n converges by Newtons Method to p.

Step 1: "Show g is continuous at p"

Step 2: Show that $g'(x)$ small near p

Step 3: Show that g maps I to I

Q#1: What is the order of convergence

A STEP ONE - Show that $g(x)$ is continuous at $p(x)$

Given: ① f is twice differentiable on $(a,b) \Rightarrow$
f is continuous on (a,b)

② $g(x) = x - \frac{f(x)}{f'(x)}$

③ $f'(p) \neq 0$

Proof ① only continuity issue is $f'(x) = 0$

② however $f'(p) \neq 0$ (given)

③ Therefore due to continuity of $f'(x)$
$\exists \delta_1 \Rightarrow f'(x) \neq 0$ for $x \in [p-\delta_1, p+\delta_1] \subset [a,b]$

④ $\therefore g(x)$ is continuous on $I_1 = [p-\delta_1, p+\delta_1]$

QED

B STEP TWO Show that $g'(x)$ is small near p (i.e. $|g'(x)| < 1$ near p)

① $g(x) = x - \frac{f(x)}{f'(x)}$

$\Rightarrow g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2}$ Quotient Rule

$= \cancel{1-1} + \frac{f(x)f''(x)}{[f'(x)]^2} \Rightarrow \boxed{g'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}}$

② since $f(p) = 0 \Rightarrow g'(p) = 0$

③ $g'(x)$ is continuous since $f(x)$, $f'(x)$, $f''(x)$ all continuous & $f'(x) \neq 0$ (given)

④ Therefore $\exists \delta_2$ such that $|g'(x)| < 1$ for $x \in [p-\delta_2, p+\delta_2]$

⑤ $\therefore |g'(x)| < 1$ on $I_2 = [p-\delta_2, p+\delta_2] \subset [a,b]$

Step 3 Show that g maps I to I

① Let $x \in I = [p+\delta, p-\delta]$

② Then

$$|g(x) - p| = |g(x) - g(p)|$$

$$= g'(\xi)|x-p| \quad \text{Int Val Theorem}$$

for ξ between x & p

$$\leq k|x-p| < |x-p| \leq \delta$$

③ Therefore: $|g(x) - p| \leq \delta$

$$\Rightarrow -\delta \leq g(x) - p \leq \delta$$

$$\Rightarrow p-\delta \leq g(x) \leq p+\delta$$

④ $\therefore$ $g(x) \in I = [p+\delta, p-\delta]$

$\Rightarrow$ Hence we have fulfilled the conditions of FP Theorems, thus an FP algorithm given by

$$g(x) = x - \frac{f(x)}{f'(x)}$$

will converge to a F.P. 'p' for some δ neighborhood around 'p'.

Q1: What is the order of convergence?

① In Step 2 we showed that $g'(x) = \dfrac{f(x)\, f''(x)}{[f'(p)]^2}$

$\Rightarrow$ since $f(p) = 0 \Rightarrow g'(p) = 0$ $\therefore$ order of convergence is at least 2

VI Order of Convergence for Roots w/ Multiplicity > 1

Let $f(x) = (x-p)^m q(x)$

① $g(x) = x - \dfrac{f(x)}{f'(x)} = x - \dfrac{(x-p)^m q(x)}{m(x-p)^{m-1} q(x) + (x-p)^m q'(x)}$

$= x - \dfrac{(x-p)q(x)}{mq(x) + (x-p)q'(x)}$

② $g'(x) = 1 - \dfrac{[mq(x) + (x-p)q'(x)][q(x) + (x-p)q'(x)] - (x-p)(q(x))[mq'(x) + q'(x) + (x-p)q''(x)]}{[mq(x) + (x-p)q'(x)]^2}$

$\Rightarrow g'(p) = 1 - \dfrac{mq(x)(q(x))}{m^2 q^2(x)} = 1 - \dfrac{1}{m}$

③ $\therefore g'(p) = 0$ only if $m = 1$

④ $\therefore m > 1 \quad g'(p) \neq 0 \quad \therefore$ O.O.C is linear

> In General: For Any Root $m > 2$, Newtons method is Slower than Bisection

Not Proven

② Now evaluate $g''(x)$

$$\Rightarrow g''(x) = \frac{d}{dx}\left[\frac{f(x)f''(x)}{[f'(x)]^2}\right]$$

$$= \Big([f'(x)]^2\,(f'(x)f''(x) + f(x)f'''(x))$$
$$- f(x)f''(x)\,(2)(f'(x))(f''(x))\Big)\Big/(f'(x))^4$$

$$= \frac{f''(x)}{f'(x)} + \frac{f(x)f'''(x)}{(f'(x))^2} - \frac{2f(x)(f''(x))^2}{f'(x)^3}$$

$$\Rightarrow g''(p) = \frac{f''(p)}{f'(p)} \Leftarrow \text{generally this is } \neq 0$$

③ ∴ Newtons method has order of convergence $\underline{\underline{=2}}$

In General for Newton's Method:

$$\boxed{\lim_{n\to\infty} \frac{|p_n - p|}{|p_{n-1} - p|^2} = \frac{f''(p)}{f'(p)}}$$

→ NOT PROVEN IN BOOK