Lecture Notes

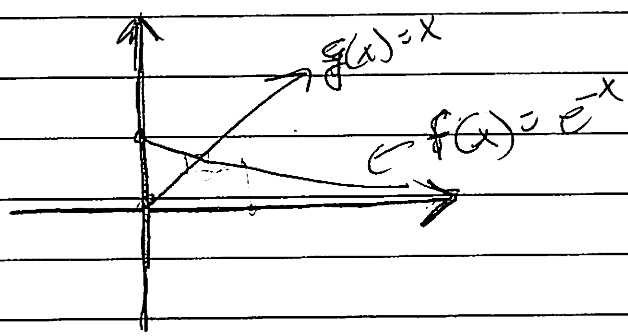
CHAPTER 2.3
Fixed Point Iteration Schemes

I **Definition:** "A Fixed Point" of a function $f(x)$ is any point p such that

$$f(p) = p$$

A **Example:** let $f(x) = e^{-x}$, is there a 'p' such that $e^{-p} = p$

Construct a graph of $f(x) = e^{-x}$, $g(x) = x$



clearly we see there exists a point such that $e^{-x} = x$

⇒ Can we set up an iterative scheme to calculate this:

i.e. $p_{n+1} = e^{-p_n}$, guess $p_0 = 1$

n	p	p_n	p_{n+1}	e_n	$\frac{e_{n+1}}{e_n}$
0		1	.367879		
⋮		⋮			⋮
20			.567141	w/ ooc	order 1

B. Some other experiments

(1) $e^x \Rightarrow$ should I even bother

max x

min x guess

(2) a) $1 - \frac{1}{2}x^2 = x$ (nice tight spiral) $[-1, +1, .5, .0001, 250]$

b) $1 - \frac{3}{4}x^2 = x$ (spiral not so tight tol

c) $1 - \frac{9}{10}x^2 = x$ (2 cycle)

d) $1 - x^2 = x$ (2 cycle)

e) $1 - 1.5x^2 = x$ (4 cycle)

f) $1 - 1.95x^2 = x$ (chaos)

g) $1 - 2x^2 = x$ (lucky bingo

h) $1 - 2.1x^2 = x$

C. Turning Root Searches into Fixed Pt Searches

$$x^3 + x^2 - 3x - 3 = 0$$

$$x = g_1(x) = \frac{x^3 + x^2 - 3x}{3} \quad \boxed{\text{go 1}} \qquad x = g_5(x) = \left(\frac{3x + 3 - x^2}{x}\right)^{1/2}$$

$$x = g_2(x) = -1 + \frac{3x + 3}{x^2} \quad \boxed{\text{go 2}}$$

$$x = g_3(x) = \sqrt[3]{3 + 3x - x^2} \quad \boxed{\text{go 3}}$$

$$x = g_4(x) = \sqrt[2]{3x + 3 - x^3}$$

II Now Some Analysis

A Recall: The Mean Value Theorem

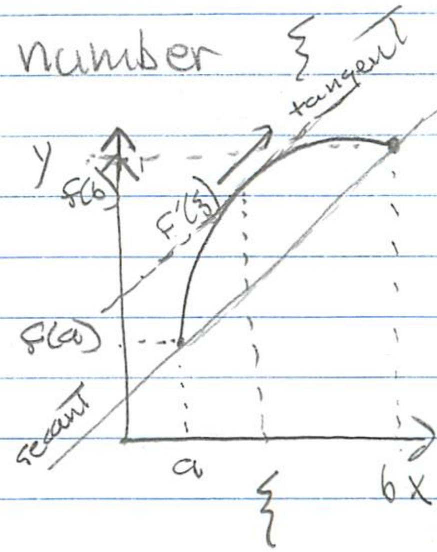
If: ① $f(x)$ is continuous on $[a,b]$
② $f(x)$ is differentiable on (a,b)

Then: There exist a real number ξ on (a,b) such that

$$f'(\xi) = \frac{f(b)-f(a)}{b-a}$$

i.e. slope of tangent line

slope of secant line



or $f(b)-f(a) = f'(\xi)(b-a)$

B. A "Bag of Theorems"

Given: ① g is continuous on $[a,b]$

② $g : [a,b] \to [a,b]$ "really restrictive"

③ g is differentiable on (a,b)

④ $|g'(x)| < 1$ for $x \in (a,b)$ "really, really restrictive"

Then ⓐ a fixed point exists & is unique

$\boxed{\text{Proof p.85}}$

ⓑ a sequence $\{p_n\}$ generated by $p_n = g(p_{n-1})$ converges to a fixed point p for any $p_0 \in [a,b]$

$\boxed{\text{Proof p.89}}$

ⓒ $|p_n - p_{n-1}| < k^n \max(p_0 - a, p_0 - b)$

here $k = |g'(x)| < 1$

$\boxed{\text{Proof p.90}}$

ⓓ If $g'(p) \neq 0$, then for any $p_0 \in [a,b]$ the sequence p_n converges linearly

but

i.e. $\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lambda \in (0,1)$

ⓔ If $g'(p) = g''(p) \ldots g^{\alpha-1}(p) = 0$ then

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^\alpha} = \frac{|g^{(\alpha)}(p)|}{\alpha!}$$