Lecture Notes

Chapter 2.1
Rootfinding / Bisection METHOD

I GENERAL PROBLEM:

Given a function $f(x)$ find a value for x, such that $f(x) = 0$

II DEFINITION: Multiplicity

A root p of the equation $f(x)=0$ is said to be:

"A ROOT OF MULTIPLICITY m"

If it can be written as: $f(x) = (x-p)^m q(x)$

A Example:

$$x^5 + 4x^4 + 4x^3 = x^3 (x+2)^2$$

$\therefore$ $p=0$ has multiplicity 3
$p=-2$ has multiplicity 2

B Theorem:

Let: ① $f(x)$ be a continuous function
② with m continuous derivatives

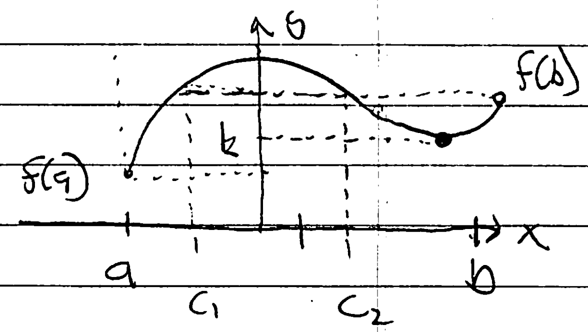
Then: $f(x)=0$ has a root of multiplicity 'm' at 'p'
If: $f(p) = f'(p) = f''(p) \ldots f^{(m-1)}(p) = 0$
But: $f^{(m)}(p) \neq 0$

II. The BISECTION Method

A. Intermediate Value Theorem

Let: ① $f(x)$ be continuous on closed interval $[a,b]$
② k be any real # between $f(a)$ & $f(b)$

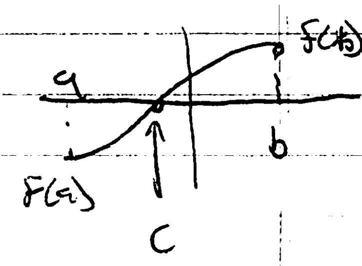
Then: $\exists$ a real number c with $a<c<b$
"there exists"
Such That: $f(c)=k$



note: two possibilities for this example

B. This is the key for the bi-section method:

If $f(a) * f(b) < 0$, there is a value c such that $f(c) = 0$



C. Algorithm (Do Example)

See page 65 ⇒ write code.

Find ROOTS FOR x^2+6x+1 $\mid \epsilon < 10^{-9}$

D. Stopping Conditions (all work equally well)

① Absolute Error $|p_n - p| < \epsilon$
② Relative Error $|p_n - p| < \epsilon p_n$
③ test for root $f(p_n) < \epsilon$

III Bisection Method - Convergence Analysis

A Theorem:

① Let f be continuous on $[a,b]$

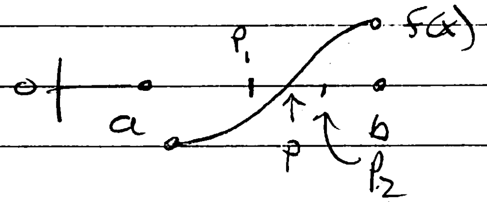
② Suppose $f(a)\,f(b) < 0$

Then the bisection methop generates a sequence p_n the converges to $p \in (a,b)$ with proberty

$$|p_n - p| = \frac{b-a}{2^n}$$

(i.e rate of convergence $= O\left(\frac{1}{2^n}\right)$)

Proof:



① Initial Guess $|p_n - p| = |b-a|$

② 1^{st} Iteration $|p_1 - p| \leq \frac{|b-a|}{2}$ "bisect"

③ 2^{nd} Iteration $|p_2 - p| \leq \frac{|b-a|/2}{2} = \frac{|b-a|}{2^n}$

⋮

④ assume after n^{th} iteration

$$|p_n - p| \leq \frac{|b-a|}{2^n}$$

⑤ then after $n+1$ iterations

$$|p_{n+1} - p| \leq \frac{|b-a|/2^n}{2} = \frac{|b-a|}{2^{n+1}}$$

Induction Proof
① Prove Base Cases, i.e. $k=0,1$
② Assume case for $k=n$
③ Prove that case for $k=n+1$ is true base on $k=n$ assumption
QED

QED By Induction