Section 1.2b
Order of Convergence
Lecture Notes

I FIRST: A Taylor Series Example

① Find Taylor Series for $\sqrt{x}$ centered on $x_0 = 4$. Expand to 3 Terms.

n	$f^{(n)}(x)$	$f^{(n)}(4)$	$f^{n}(4)/n!$
0	$x^{1/2}$	2	2
1	$-\frac{1}{2}x^{-1/2}$	$-1/4$	$-1/24$
2	$+\frac{1}{4}x^{-3/2}$	$1/32$	$1/64$
3	$-\frac{3}{8}x^{-5/2}$	$-3/256$	$-1/512$

$$\therefore \sqrt{x} \sim 2(x-4)^0 - \frac{1}{4}(x-4) + \frac{1}{64}(x-4)^2$$

$$= \boxed{\sqrt{x} = 2 - \frac{1}{4}(x-4) + \frac{1}{64}(x-4)^2}$$

② What is remainder term

$$R_3 = \left|\frac{f^{(3)}(\xi)}{3!}(x-4)^3\right| = \left|\frac{-\frac{3}{8}\xi^{-5/2}}{6}\right| \left|(x-4)^3\right|$$

$$= \frac{1}{16}\xi^{-5/2}\left|(x-4)^3\right|$$

③ Suppose I am approximating square roots for $3.8 \le 4 \le 4.2$. What is biggest possible R_n

Recall $3.8 \le \xi \le 4.2$

note $\xi^{-5/2}$ is biggest w/ $\xi = 3.8$

$$R_n < \frac{1}{16}(3.8)^{-5/2}\left|(3.8-4)^3\right| = 1.776 \times 10^{-5}$$

II) Recall Definition of "Rate of Convergence"

* If $\lim_{n\to\infty} p_n = p$
* and $|p_n - p| < \lambda B_n$
* then Rate of Convergence is $O(B_n)$

III Definition For "Order of Convergence"

* If: $\lim_{n\to\infty} p_n = p$
* let $e_n = p_n - p$
* If there exists positive constants λ & α
* such that $\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^\alpha} = \lambda$
* Then $\{p_n\}$ is said to converge to p w/ 'order' α & assymptotic error λ

IV Example: Find Order of Convergence For Recursive Square Root Scheme

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

$\Rightarrow$ we know $\lim_{n \to \infty} x_n = \sqrt{a}$

$$\therefore \quad x_{n+1} - \sqrt{a} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) - \sqrt{a}$$

$$= \frac{x_n}{2} + \frac{a}{2x_n} - \sqrt{a}$$

$$= \frac{x_n^2 - 2x_n\sqrt{a} + a}{2x_n}$$

$$\Rightarrow \quad x_{n+1} - \sqrt{a} = \frac{(x_n - \sqrt{a})^2}{2x_n}$$

$$\Rightarrow \quad \frac{x_{n+1} - \sqrt{a}}{(x_n - \sqrt{a})^2} = \frac{1}{2x_n}$$

$$\Rightarrow \quad \lim_{n \to \infty} \left| \frac{x_{n+1} - \sqrt{a}}{(x_n - \sqrt{a})^{\alpha}} \right| = \lim_{n \to \infty} \frac{1}{2x_n} = \underbrace{\frac{1}{2\sqrt{a}}}_{\lambda}$$

$\therefore$ ORDER OF CONVERGENCE EQUALS 2 WITH ASYMPTOTIC ERROR CONSTANT $1/2\sqrt{a}$

Example: Numerical Verification of Previos Example

Find $\sqrt{4}$ w/ initial guess $x_0 = 13$

$\boxed{\text{EXCEL CHART}}$

n	x_n	$\|x_n-2\|$	$\|x_{n+1}-2\|/\|x_n-2\|$	$\|x_{n+1}-2\|/\|x_n-2\|^2$	$\|x_{n+1}-2\|/\|x_n-2\|^3$
0	13	11			
1	6.65384615	4.65384615	0.423076923	0.038461538	0.003496503
2	3.62750111	1.62750111	0.349710983	0.075144509	0.016146754
3	2.36509429	0.36509429	0.224328134	0.137835933	0.08469176
4	2.02817939	0.02817939	0.077183876	0.211408062	0.579050588
5	2.00019576	0.00019576	0.006946967	0.246526516	8.748468204
6	2.00000001	9.5797E-09	4.89355E-05	0.249975525	1276.940346
7	2	0	0	0	0

Looks like! $\lim_{n\to\infty} \frac{|x_{n+1}-2|}{|x_n-2|^2} = .25$

note: that $\lim_{n\to\infty} \frac{|x_{n+1}-2|}{|x_n-2|} = 0$ but this is not a "positive" constant

note: $\lim_{n\to\infty} \frac{|x_{n+1}-2|}{|x_n-2|^3}$ DNE

$\therefore$ ORDER OF CONVERGENCE IS 2 w/ ASYMPTOTIC ERROR .25

NOTE
$\frac{1}{2\sqrt{a}} = .25$ IF $a=4$