Section 1.2
Rate of Convergence
Lecture Notes

I. Definition of Convergence

A sequence $\{x_n\}$ converges to a value L

If... $\boxed{\lim_{x\to\infty} x_n = L}$ "Calc II Stuff"

or... $\boxed{\lim_{x\to\infty} |x_n - L| = 0}$

II Definition: Rate of Convergence

Assume: (1) $\{p_n\}$ converges to p

If: (1) there exists a sequence $\{B_n\}$
(2) and a 'λ' independent of 'n'

Such that: $|p_n - p| \leq \lambda |B_n|$

Then We Say: "$\{p_n\}$ converges to p w/ Rate of convergence $O(B_n)$"

"big O of B_n"

III **Example:** Find Rate of Convergence of $\left\{ \frac{n+3}{n+7} \right\}$ ← 'p_n'

$$\lim_{n \to \infty} \frac{n+3}{n+7} = 1 \leftarrow 'p'$$

$$|p_n - p| = \left| \frac{n+3}{n+7} - 1 \right| = \left| \frac{n+3-n-7}{n+7} \right| = \left| \frac{-4}{n+7} \right| < \underbrace{4}_{'\lambda'} \cdot \underbrace{\frac{1}{n}}_{\beta_n}$$

$\therefore$ $\boxed{\text{Rate of Convergence is } O\left(\frac{1}{n}\right)}$

IV **Example:** Find Rate of Convergence of $\left\{ \frac{2^n+3}{2^n+7} \right\}$

$$\lim_{n \to \infty} \left\{ \frac{2^n+3}{2^n+7} \right\} = 1$$

$$|p_n - p| = \left| \frac{2^n+3}{2^n+7} - 1 \right| = \left| \frac{2^n+3-2^n-7}{2^n+7} \right| = \left| \frac{-4}{2^n+7} \right| < \underbrace{4}_{'\lambda'} \cdot \underbrace{\frac{1}{2^n}}_{'\beta_n'}$$

$\therefore$ $\boxed{\text{Rate of Convergence is } O\left(\frac{1}{2^n}\right)}$

*** Based on these results which sequences approaches '1' faster?

V **Example:** Find ROC of $\left\{ \frac{\sin(n)}{n} \right\}$

$$\lim_{n \to \infty} \frac{\sin(n)}{n} = \underbrace{0}_{p} \quad \text{by squeeze theorem}$$

$$|p_n - p| = \left| \frac{\sin(n)}{n} - 0 \right| < \left| \frac{1}{n} \right| = \frac{1}{n}$$

Rate of Convergence $O\left(\frac{1}{n}\right)$

IV Recall Taylor Series

Any continuous function $f(x)$ whose derivatives are also continuous on interval (a,b) can be written as Taylor Series "centered on x_0"

$$\boxed{f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}$$

a) Some Common Taylor Series (centered on $x_0=0$)

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \ldots$$

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots$$

(more on page 27)

b) If I truncate a Taylor Series after n term I induce an error

$$\therefore P_n = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

$\Rightarrow$ where 'ξ' is some # in interval of interest

Remainder Term

Example - ROC of A Function

VI. Find ROC of $\frac{\sin(x)}{x}$ as $x \to 0$

$\Rightarrow$ note that $\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1$

"L'Hospitals Rule"

$$\left| \frac{\sin(x)}{x} - 1 \right| = \left| \frac{\sin(x) - x}{x} \right|$$

$$P_3 = x - \frac{1}{3!} x^3 \sin(\xi)$$

First order Taylor Series Poly of $\sin(x)$

$$\Rightarrow \left| \frac{\sin(x) - x}{x} \right| = \left| \frac{\cancel{x} - \frac{1}{3!} x^3 \sin(\xi) \cancel{- x}}{x} \right| = \frac{1}{6} x^2 \sin(\xi) < \frac{1}{6} x^2$$

$\therefore$ $\boxed{\text{Rate of Convergence is } O(x^2)}$