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SM365 – Equations for Final Exam

Rate of convergence-Sequences Let $\{p_n\}$ be a sequence that converges to a number p. If there exists a sequence $\{\beta_n\}$ that converges to zero and a positive constant λ, independent of n, such that

$$|p_n - p| \le \lambda |\beta_n|$$

for all **sufficiently large values of** n, then $\{p_n\}$ is said to converge to p with RATE OF CONVERGENCE $O(\beta_n)$. (*Note:* The sequence $\{\beta_n\}$ is typically of the form $1/n^a$ or $1/a^n$.)

Rate of convergence-Functions Let f be a function defined on the interval (a, b) that contains $x = 0$, and suppose $\lim_{x\to 0} f(x) = L$. If there exists a function g for which $\lim_{x\to 0} g(x) = 0$ and a positive constant K such that

$$|f(x) - L| \le K|g(x)|$$

for all **sufficiently small values of** x, then $f(x)$ is said to converge to L with RATE OF CONVERGENCE $O(g(x))$. (*Note:* The function $g(x)$ is typically of the form x^a where $a > 0$.)

Order of convergence-Sequences Let $\{p_n\}$ be a sequence that converges to a number p. Let $e_n = p_n - p$ for $n \ge 0$. If there exist positive constants λ and α such that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^\alpha} = \lambda,$$

then $\{p_n\}$ is said to converge to p of ORDER OF CONVERGENCE α with ASYMPTOTIC ERROR CONSTANT λ.

Taylor's Theorem Suppose that f is continuous on $[a, b]$, has continuous derivatives on (a, b) and $f^{(n+1)}$ exists on $[a, b]$. Let $x_0 \in [a, b]$. For every $x \in [a, b]$, there exists a number $\xi(x)$ (ξ depends on x) between x and x_0 such that

$$f(x) = P_n(x) + R_n(x),$$

where P_n is the nth degree Taylor polynomial

$$P_n = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

and R_n is the remainder term

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

Floating Point Number System $\mathbf{F}(\beta, k, m, M)$ is a subset of the real number system characterized by the parameters

β	the base
k	the number of digits
m	the minimum exponent
M	the maximum exponent

Elements of $\mathbf{F}(\beta, k, m, M)$ are those real numbers that can be expressed exactly as

$$\pm(0.d_1 d_2 \ldots d_k)_\beta \times \beta^e, \quad m \le e \le M.$$

The first digit, d_1, must be non-zero, except when representing zero.

Errors Let p^* denote any approximation to the value p. The ABSOLUTE ERROR in p^* is given by

$$|p - p^*|$$

The RELATIVE ERROR is

$$\frac{|p - p^*|}{|p|}$$

usually expressed as percentage.

Significant digits Suppose that $x \ne 0$ and y are nearly equal and

$$\beta^{-(t+1)} < \left|\frac{x - y}{x}\right| \le \beta^t,$$

for some positive integer t. The we say that x and y agree to at least t and at most $t + 1$ SIGNIFICANT base β DIGITS.

Multiplicity If f is continuous and has m continuous derivatives. The equation $f(x) = 0$ has a root of multiplicity m at $x = p$ iff

$$f(p) = f'(p) = f''(p) = \ldots = f^{(m-1)}(p) = 0$$

but $f^{(m)}(p) \neq 0$.

Intermediate Value Theorem Let f be a continuous function over the interval $[a, b]$, and k be any real number that lies between $f(a)$ and $f(b)$. Then there exists a real number c with $a < c < b$ such that $f(c) = k$.

Bisection Method Let f be a continuous function over the interval $[a, b]$, and suppose that $f(a)f(b) < 0$. The bisection method generates a sequence of approximations $\{p_n\}$ which converges to a root $p \in (a, b)$ with the property

$$|p_n - p| \leq \frac{b-a}{2^n}$$

Mean Value Theorem If f is a continuous function over the interval $[a, b]$, and differentiable over the interval (a, b), then there exists a number $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Existence and Uniqueness of Fixed point Let g be continuous on the closed interval $[a, b]$ with $g : [a, b] \to [a, b]$. Then g has a fixed point $p \in [a, b]$. Furthermore, if g is differentiable on the open interval (a, b) and there exists a positive constant $k < 1$ such that $|g'(x)| \leq k < 1$ for all $x \in (a, b)$, then the fixed point is unique.

Convergence of Fixed point iteration Let g satisfy all requirements of the Existence and Uniqueness of a fixed point theorem. Then

(1) the sequence $\{p_n\}$ generated by $p_n = g(p_{n-1})$ converges to the fixed point p for any $p_0 \in [a, b]$;

(2) $|p_n - p_{n-1}| \leq k^n \max(p_0 - a, b - p_0)$; and

(3) $|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|$

Order of Convergence of Fixed point iteration Let g be continuous on the closed interval $[a, b]$ with $g : [a, b] \to [a, b]$. Then g has a fixed point $p \in [a, b]$. Furthermore, if g is differentiable on the open interval (a, b) and there exists a positive constant $k < 1$ such that $|g'(x)| \leq k < 1$ for all $x \in (a, b)$. If $g'(p) \neq 0$, then for any $p_0 \in [a, b]$ the sequence $p_n = g(p_{n-1})$ converges only linearly to the fixed point p.

Stopping conditions

(i) Linear Convergence

$$g'(p) \approx \frac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}}$$

$$|e_n| \approx \left|\frac{g'(p)}{g'(p) - 1}\right| |p_n - p_{n-1}| \leq TOLERANCE$$

(ii) Superlinear Convergence

$$|e_n| = |p_{n-1} - p| \approx |p_n - p_{n-1}| \leq TOLERANCE$$

Newton's Method is the fixed point iteration scheme based on the iteration function

$$g(x) = x - \frac{f(x)}{f'(x)};$$

that is, starting from an initial approximation, p_0, the sequence $\{p_n\}$ is generated via $p_n = g(p_{n-1})$.

Order of Convergence of Newton's Method Provided $f'(p) \neq 0$, convergence is at least quadratic. If $f'(p) = 0$ convergence is linear.

Stopping conditions Since we have superlinear convergence:

$$|p_{n-1} - p| \approx |p_n - p_{n-1}| \leq TOLERANCE$$

Singular/Non-singular Matrix Let A be an $n \times n$ matrix. If there exists a **non-zero** vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$A\mathbf{x} = \mathbf{0}$$

then A is called a **singular** matrix. If there is no such $\mathbf{x}$, then the matrix is called **non-singular**.

Vector Norm The function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is called a vector norm if for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$, the following properties hold:

(i) $\|\mathbf{x}\| \geq 0$;

(ii) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$;

(iii) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$; and

(iv) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

l_2-norm of a vector The l_2-norm of a vector $\mathbf{x} \in \mathbb{R}^n$, which is denoted by $\|\mathbf{x}\|_2$ is defined by

$$\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}.$$

l_∞-norm of a vector The l_∞-norm of a vector $\mathbf{x} \in \mathbb{R}^n$, which is denoted by $\|\mathbf{x}\|_\infty$ is defined by

$$\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

Cauchy-Buniakowski-Schwartz Inequality Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then:

$$\left|\sum_{i=1}^{n} x_i y_i\right| \leq \|\mathbf{x}\|_2 \|\mathbf{y}\|_2.$$

Norm Equivalence Let $\|\cdot\|$ and $\|\cdot\|'$ be vector norms on $\mathbb{R}^n$. If there exist positive constants c_1 and c_2 such that

$$c_1\|\mathbf{x}\| \leq \|\mathbf{x}\|' \leq c_2\|\mathbf{x}\|$$

for all $\mathbf{x} \in \mathbb{R}^n$, the two norms are said to be equivalent.

Matrix Norm The function $\|\cdot\| : \mathbb{R}^{n\times n} \to \mathbb{R}$ is called a matrix norm if for all $A, B \in \mathbb{R}^{n\times n}$ and all $\alpha \in \mathbb{R}$, the following properties hold:

(i) $\|A\| \geq 0$;

(ii) $\|A\| = 0$ if and only if $A = 0$;

(iii) $\|\alpha A\| = |\alpha|\|A\|$;

(iv) $\|A + B\| \leq \|A\| + \|B\|$; and

(v) $\|A\,B\| \leq \|A\|\;\|B\|$

Natural Norm Let $\|\cdot\|_v$ be a vector norm. The real-valued function $\|\cdot\|$ that is defined for all $A \in \mathbb{R}^{n\times n}$ by

$$\|A\| = \max_{\|\mathbf{x}\|_v \neq \mathbf{0}} \frac{\|A\mathbf{x}\|_v}{\|\mathbf{x}\|_v},$$

is called the natural, or operator norm associated with the vector norm $\|\cdot\|_v$.

Theorem Let $\|\cdot\|_v$ be a vector norm. The natural norm associated with $\|\cdot\|_v$ is a matrix norm.

l_2-norm of a matrix The l_2-norm of a matrix $A \in \mathbb{R}^{n\times n}$, which is denoted by $\|A\|_2$ is defined by

$$\|A\|_2 = \sqrt{\rho(A^T A)},$$

where $\rho(B) = \max_{\lambda\in\sigma(B)} |\lambda|$.

l_∞-norm of a matrix The l_∞-norm of a matrix $A \in \mathbb{R}^{n\times n}$, which is denoted by $\|A\|_\infty$ is defined by

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|,$$

where $\rho(B) = \max_{\lambda\in\sigma(B)} |\lambda|$.

Error estimate Theorem Let A be a non-singular matrix, $\tilde{\mathbf{x}}$ be an approximate solution to the linear system $A\mathbf{x} = \mathbf{b}$, $\mathbf{r} = A\tilde{\mathbf{x}} - \mathbf{b}$ and $\mathbf{e} = \tilde{\mathbf{x}} - \mathbf{x}$. Then, for any natural matrix norm $\|\cdot\|$,

$$\frac{1}{\|A\|}\|\mathbf{r}\| \leq \|\mathbf{e}\| \leq \|A^{-1}\|\|\mathbf{r}\|$$

and

$$\frac{1}{\|A\|\;\|A^{-1}\|}\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} \leq \frac{\|\mathbf{e}\|}{\|\mathbf{x}\|} \leq \|A\|\;\|A^{-1}\|\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}.$$

Condition number Let $A \in \mathbb{R}^{n\times n}$ be non-singular; then the condition number is

$$\kappa(A) = \|A\|\|A^{-1}\|.$$

Perturbations to A and $\mathbf{b}$ Let δA and $\delta\mathbf{b}$ denote the perturbations to A and $\mathbf{b}$ respectively, and let $\mathbf{x} + \delta\mathbf{x}$ denote the solution to the system

$$(A + \delta A)(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b} + \delta\mathbf{b}.$$

Suppose that $\|\delta A\| < 1/\|A^{-1}\|$, which guarantees that $A + \delta A$ remains nonsingular. Then

$$\frac{\|\delta \mathbf{x}\|}{\|\mathbf{x}\|} \leq \frac{\kappa(A)}{1 - \kappa(A)\frac{\|\delta A\|}{\|A\|\|}} \left(\frac{\|\delta \mathbf{b}\|}{\|\mathbf{b}\|} + \frac{\|\delta A\|}{\|A\|\|} \right).$$

Iterative Techniques Solving $A\mathbf{x} = \mathbf{b}$ can be solved as a fixed point problem via the method:

$$\mathbf{x}^{(k+1)} = T\mathbf{x}^{(k)} + \mathbf{c}$$

Theorem Let A be an $n \times n$ matrix. Then the following are equivalent:

1. $\rho(A) < 1$, where $\rho(A)$ denotes the spectral radius of A
2. $A^k \to 0$ as $k \to \infty$
3. $A^k\mathbf{x} \to \mathbf{0}$ as $k \to \infty$ for any vector $\mathbf{x}$.

Error Evolution Provided $\|T\| < 1$

$$\|\mathbf{x}^{(k)}\| \leq \frac{\|T\|^k}{1 - \|T\|} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|$$

Consistency Equation For any fixed point iteration, the matrix T and vector $\mathbf{c}$ must satisfy:

$$(I - T)^{-1}\mathbf{c} = A^{-1}\mathbf{b}.$$

Jacobi Method Let $A = D - L - U$, where D is the diagonal, $-L$ the lower triangular, and $-U$ is the upper triangular parts of the matrix. Then we have

$$T_{jac} = D^{-1}(L + U) \quad \text{and} \quad \mathbf{c}_{jac} = D^{-1}\mathbf{b}$$

or

$$x_i^{(k+1)} = \frac{1}{a_{i,i}} \left[b_i - \sum_{j=1}^{i-1} a_{i,j} x_j^{(k)} - \sum_{j=i+1}^{n} a_{i,j} x_j^{(k)} \right]$$

Gauss-Seidel With A, D, $-L$, and $-U$ as above we have

$$T_{gs} = (D - L)^{-1}U \quad \text{and} \quad \mathbf{c}_{gs} = (D - L)^{-1}\mathbf{b}$$

or

$$x_i^{(k+1)} = \frac{1}{a_{i,i}} \left[b_i - \sum_{j=1}^{i-1} a_{i,j} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{i,j} x_j^{(k)} \right]$$

SOR With A, D, $-L$, and $-U$ as above and $0 < \omega < 2$, we have

$$T_{sor} = \left(\frac{1}{\omega} D - L \right)^{-1} U$$

and

$$\mathbf{c}_{sor} = \left(\frac{1}{\omega} D - L \right)^{-1} \mathbf{b}$$

or

$$x_i^{(k+1)} = (1 - \omega)x_i^{(k)} + \frac{\omega}{a_{i,i}} \left[b_i - \sum_{j=1}^{i-1} a_{i,j} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{i,j} x_j^{(k)} \right]$$

Specific Convergence Properties

1. If A is real and symmetric with all positive diagonal elements, then the Gauss-Seidel method converges if and only if A is positive definite (i.e. $\mathbf{x}^T A\mathbf{x} > 0$ for any non-zero vector $\mathbf{x}$.)
2. If A is positive definite , then the Gauss-Seidel method will converge for any choice of the initial vector $\mathbf{x}^{(0)}$.

Theorem

1. If A has all non-zero diagonal elements, then $\rho(T_{sor}) \geq |\omega - 1|$. Therefore, the SOR method can converge only if $0 < \omega < 2$.
2. If A is positive definite and $0 < \omega < 2$, then the SOR method will converge for any choice of the initial vector $\mathbf{x}^{(0)}$.

Theorem Suppose that A is an $n \times n$ matrix. If $a_{i,i} > 0$ for each i and $a_{i,j} \leq 0$ whenever $i \neq j$, then one and only one of the following statements holds:

1. $0 \leq \rho(T_{gs}) = \rho(T_{jac}) < 1$;
2. $1 < \rho(T_{jac}) < \rho(T_{gs})$;
3. $\rho(T_{jac}) = \rho(T_{gs}) = 0$;
4. $\rho(T_{gs}) = \rho(T_{jac}) = 1$.

Fundamental property of polynomial interpolation An interpolating polynomial, P_n, of degree at most n satisfies

$$P_n(x_i) = f(x_i),$$

where $x_0, x_1, \ldots, x_n$ are distinct though not necessarily uniform.

Lagrange Interpolation For $i = 0, 1, 2, \ldots, n$ define the nth degree polynomials:

$$L_{n,i} = \frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)} = \prod_{i=0, i\neq j}^{n} \frac{x-x_j}{x_i-x_j}.$$

Then the nth degree Lagrange interpolating polynomial is:

$$P_n(x) = f(x_0)L_{n,0} + f(x_1)L_{n,1} + \cdots f(x_n)L_{n,n}$$

Uniqueness Theorem If $x_0, x_1, \ldots, x_n$ are $n+1$ distinct points and f is defined at those points, then there exists a unique polynomial, P, of degree at most n such that P interpolates f.

Interpolation Error If $x_0, x_1, \ldots, x_n$ are $n+1$ distinct points in $[a, b]$ and f is continuous on $[a, b]$ and has $n+1$ continuous derivatives on $[a, b]$, then for each $x \in [a, b]$ there exists a $\xi(x) \in [a, b]$ such that

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\cdots(x-x_n).$$

Newton form of the interpolating polynomial:

$$\begin{aligned} P_{0,1,2,\ldots,n}(x) = a_0 &+ a_1(x-x_0) \\ &+ a_2(x-x_0)(x-x_1) + \cdots \\ &+ a_n(x-x_0)\cdots(x-x_{n-1}), \end{aligned}$$

where

$$a_k = f[x_0, \ldots x_k].$$

(see below for definition)

Divided Differences Let f be a function defined at the distinct points $x_0, x_1, \ldots x_n$. The zeroth divided difference of f with respect to the point x_i is

$$f[x_i] = f(x_i).$$

For $0 < k \leq n$, the kth divided difference of f with respect to the points $x_i, x_{i+1}, \ldots, x_{i+k}$ is

$$f[x_i, x_{i+1}, \ldots, x_{i+k}] = \frac{f[x_{i+1}, \ldots, x_{i+k}] - f[x_i, x_{i+1}, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}$$

Interpolation Error Let $x_0, x_1, \ldots, x_n$ be $n+1$ distinct points in $[a, b]$. If f is continuous on $[a, b]$ and has n continuous derivatives on (a, b), then there exists a $\xi \in (a, b)$ such that

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

Numerical Differentiation Below are standard formulae along with their errors:

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{h}{2}f''(\xi)$$

$$f'(x_0) = \frac{f(x_0) - f(x_0-h)}{h} - \frac{h}{2}f''(\xi)$$

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} - \frac{h^2}{6}f'''(\xi)$$

$$f''(x_0) = \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} - \frac{h^2}{12}f^{(4)}(\xi)$$

Linear Regression Given $x_0, x_1, \ldots, x_n$ and $y_0, y_1, \ldots, y_n$, the line of "best-fit" is $\hat{y} = a + bx$, where

$$b = \frac{n\sum_{i=1}^{n} x_i y_i - n^2\bar{x}\bar{y}}{n\sum_{i=1}^{n} x_i^2 - (n\bar{x})^2}$$

$$a = \bar{y} - b\bar{x}$$

and

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

Richardson Extrapolation Method to obtain higher order approximations for a value of a derivative. Construct a table a follows:

Step size	$O(h^{p_1})$	$O(h^{p_2})$	$O(h^{p_3})$
h	$D_h^{(1)}$		
$h/2$	$D_{h/2}^{(1)}$	$D_h^{(2)}$	
$h/3$	$D_{h/4}^{(1)}$	$D_{h/2}^{(2)}$	$D_h^{(3)}$

The first column is constructed using a rule from the list of the previous item or other. To construct the next column use the formula:

$$\frac{2^p D_{h/2} - D_h}{2^p - 1}$$